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# Exponential stabilization of the class of the switched systems with mixed time varying delays in state and control

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**Abstract**

This paper presents the problem of exponential stabilization of switched systems with mixed time-varying delays in state and control. Based on the partitioning of the stability state regions into convex cones, a constructive geometric design for switching laws is put forward. By using an improved Lyapunov–Krasovskii functional in combination with matrix knowledge, we design a state feedback controller that guarantees the closed-loop system to be exponentially stable. The obtained conditions are given in terms of linear matrix inequalities (LMIs), which can be effectively decoded in polynomial time by various computational tools such as the LMI tool in MATLAB software. A numerical example is proposed to illustrate the effectiveness of the obtained results.

***Keywords:*** Exponential stabilization, switched systems, varying delay, lyapunov function

## 1. Introduction

Stability theory of dynamical systems was first studied by the mathematician Lyapunov in the late 19th century. Since then, Lyapunov stability theory has become an essential part of the study of differential equations, system theory, and applications.

In particular, the stability of hybrid systems has attracted a lot of attention from many researchers such as Y. Zhang [1], Z. Sun [2], and A. V. Savkin [3]. Switched systems are an important class of hybrid systems [3], [4]. Switched systems, which are a collection of subsystems and switching rules, can be described by a differential equation of the form:

$$\dot{x}(t) = f_{\sigma}(t,x),\ t \geq 0,$$

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where { $f_\sigma(): \sigma \in \bar{m}$={1, 2,…, m}} is a family of functions that is parameterized by some index set $\bar{m}$ which is typically a finite set and $\sigma(.)$ depending on the system state in each time is a switching rule, which determines a switching sequence for a given switching system.

The class of switched systems is of particular significance and has many important applications in practice. Switching systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models, such as manufacturing, communication networks, automotive engineering control, chemical processes [2], [3], [5]. As a consequence, many important and interesting results on switched systems have been reported and various issues have been studied by many authors [4], [6]–[17].

In this paper, we extend the results of [18] to switched systems with mixed delays in state and control. The switched system in the research [18] is given by formular (1):

$$\dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-\tau) + E_\sigma \int_{t-\tau}^{t} x(s)ds + B_\sigma u(t) + C_\sigma u(t-r) + F_\sigma \int_{t-r}^{t} u(s)ds \qquad (1)$$

In our research, we will consider switched system with mixed time varying delays in state and control as presented in (2):

$$\dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-h_1(t)) + E_\sigma \int_{t-k_1(t)}^{t} x(s)ds + B_\sigma u(t) + C_\sigma u(t-h_2(t)) + F_\sigma \int_{t-k_2(t)}^{t} u(s)ds, t \in \mathbb{R}^+ \qquad (2)$$

We used the Lyapunov function method with the application of the LMI tool in the MATLAB software. The goal of this research is to find the state feedback controller $u(t)$ (which will be introduced in Section 2) and the $\sigma(.)$ rule in order to apply the Lyapunov function method. From this, we obtain the theorem to be proved. We also use MATLAB to provide a numerical example to illustrate the problem.

The paper is organized as follows. Section 2 presents main concepts and lemmas needed for the proof of the main result. Exponential stabilization of the class of the switched systems with mixed time varying delays in state and control is presented, proved, and illustrated by a numerical example in Section 3. The paper ends with conclusions and the cited references.

## 2. Problem statement and preliminaries

First, we introduce some notations, concepts and lemmas which are necessary for this present work. The following notations will be used throughout this paper: $R^+$ denotes the set of all non-negative real numbers; $R^n$ denotes the $n$-dimensional Euclidean space, with the Euclidean norm $\|.\|$ and scalar product $\langle x, y\rangle = x^T y$. For a real matrix *A*, $\lambda_{max}(A)$ *and* $\lambda_{min}(A)$ denote the maximal and the minimal eigenvalue of *A*, respectively; $A^T$ denotes the transpose of the matrix *A*. $Q \geq 0$ ($Q > 0$, resp.) means $Q$ is semi-positive definite (positive definite, resp.), $A \geq B$ means $A - B \geq 0$.

Consider a switched system with the delays are varying in state and control of the form:

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-h_1(t)) + E_\sigma \int_{t-k_1(t)}^{t} x(s)ds + B_\sigma u(t) + C_\sigma u(t-h_2(t)) + F_\sigma \int_{t-k_2(t)}^{t} u(s)ds, t \in \mathbb{R}^+ \\ x(t) = \phi(t), t \in [\text{-d}, 0], \text{ d= max}\{h_1, h_2, k_1, k_2\}, \end{cases} \qquad (3)$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control, and $h_1(\mathrm{t}), h_2(\mathrm{t}),\ k_1(\mathrm{t}), k_2(t)$ are the varying delays **satisfying the condition:**

$$0\le h_1(\mathrm{t})\le h_1,\ \dot{h}_1(\mathrm{t})\le\delta_1<1,$$

$$0\le h_2(\mathrm{t})\le h_2,\ \dot{h}_2(\mathrm{t})\le\delta_2<1,$$

$$0\le k_1(\mathrm{t})\le k_1,\ 0\le k_2(\mathrm{t})\le k_2,$$

and $\sigma\in\bar{m}=\{1, 2 ,..., \mathrm{m}\}$ is a switching rule depending on time and the system state. $A_\sigma, B_\sigma, C_\sigma, D_\sigma, E_\sigma, F_\sigma(\sigma\in\bar{m})$ are constant matrices with appropriate dimensions.

Before presenting the main result, we recall some well-known concepts, remarks and lemmas which will be used in the proof.

**Definition 1** ([6]). Given $\alpha>0$. System (3) with $u(t)=0$ is $\alpha$-exponentially stable if there exists an $\sigma$ switching rule and a constant $\beta>0$ such that every solution $x(t,\phi)$ of the system satisfies:

$$\| x(t,\phi)\|\le\beta e^{-\alpha t}\,\|\|\phi\|,\ t\ge 0.$$

**Definition 2** ([6]). Given $\alpha>0$. System (3) is $\alpha$-stabilizable in the sense of exponential stability if there exists a control input $K_i\in\mathbb{R}^{m\times n}(i=1,\ 2,\dots,\ \mathrm{m})$ such that the closed-loop system**:**

$$\dot{x}(t)=[A_\sigma+B_\sigma\mathrm{K}_\sigma]x(t)+D_\sigma x(t-h_1(t))+E_\sigma\int_{t-k_1(t)}^{t}x(s)ds+E_\sigma x(t-h_2(t))+F_\sigma\int_{t-k2(t)}^{t}u(s)ds$$

is $\alpha$-exponentially stable. $u(t)=K_\sigma x(t)$ is called feedback controller.

**Definition 3** ([7]). The system of matrices $\{L_i\}, i\in\bar{m}=\{1, 2,...,\mathrm{m}\}$ is said to be strictly complete if for every $x\in\mathbb{R}^n\setminus\{0\}$ there is $i\in\bar{m}$ such that $x^T L_i x<0$.

Let us define $\Omega_i=\{x\in\mathbb{R}^n\text{: } x^T L_i x<0\}$, $i\in\bar{m}$.

It is easy to show that the system $\{L_i\}, i\in\bar{m}$ is strictly complete if and only if:

$$\bigcup_{i=1}^{m}\Omega_i=\mathbb{R}^n\setminus\{0\}. \tag{4}$$

**Remark** 1 ([7]). A sufficient condition for the strict completeness of the system $\{L_i\}$ is that there exist $\beta_\mathrm{i}\ge 0,\ \sum_{i=1}^{m}\beta_\mathrm{i}>0$ such that: $\sum_{i=1}^{m}\beta_\mathrm{i}L_i<0$.

**Lemma 1** *(Matrix Cauchy Inequality* [19]*)*. For any symmetric positive definite matrix $M\in\mathbb{R}^{n\times n}$ and $x,y\in\mathbb{R}^n$, we have:

$$2\langle x,y\rangle\le\langle Mx,x\rangle+\langle M^{-1}y,y\rangle.$$

**Lemma 2** ([20]). *For any symmetric positive definite matrix* $M \in \mathbb{R}^{n\times n}$, *scalar* $\gamma > 0$ *and vector function* $\omega$*:* $[0,\gamma] \to \mathbb{R}^n$ *such that the integrals concerned are well defined, then:*

$$\left(\int_0^{\gamma} \omega(s)ds\right)^T M \left(\int_0^{\gamma} \omega(s)ds\right) \le \gamma \int_0^{\gamma} \omega^T(s) M \omega(s) ds.$$

**Lemma 3** *(Schur Complement Theorem [19]). For any constant matrices* $X, Y, Z \in \mathbb{R}^{n\times n}$, *where* $X = X^T$, $Y = Y^T > 0$. *Then* $X - Z^T Y^{-1} Z > 0$ *if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & Y \end{bmatrix} > 0 \ \ or \ \begin{bmatrix} -Y & Z \\ Z^T & X \end{bmatrix} > 0.$$

**3. Main result**

For given $\alpha > 0, h_1 \ge 0, h_2 \ge 0, k_1 \ge 0, k_2 \ge 0$, symmetric positive definite matrices *P*, *Q*, *R*, *M* and matrices $Y_i$ *(i = 1, 2,…, m)* with appropriate dimensions, we set: $L_i = A_i P + P A_i^T + 2\alpha P + G_i + k_2 R + M, \ i \in \overline{m}$,

$$\text{where } G_i = B_i Y_i + Y_i^T B_i^T + \left(\frac{1}{1-\delta_2} e^{2\alpha h_2} C_i C_i^T + k_2 e^{2\alpha k_2} F_i F_i^T\right),$$

$$\Omega_i = \{x \in \mathbb{R}^n : x^T L_i x < 0\}, \ s_i = \{\mathrm{P}x : x \in \Omega_i\}, \ i \in \overline{m},$$

$$\overline{s}_1 = s_1, \ \overline{s}_i = s_i \setminus \bigcup_{j=1}^{i-1} \overline{s}_j, \ i = 2, 3, ..., m,$$

$$U_i = [D_i P \quad E_i P], \ H = \mathrm{diag}\left[(1-\delta_1) e^{-2\alpha h_1} Q, \frac{1}{k_1} e^{-2\alpha k_1} R\right],$$

$$\alpha_1 = \lambda_{\min}(P^{-1}), \ \mu = \sqrt{1+k_2},$$

$$\alpha_2 = \lambda_{\max}(P^{-1}) + h_1 \lambda_{\max}(P^{-1} Q P^{-1}) + \frac{1}{2} k_1^2 \lambda_{\max}(P^{-1} R P^{-1}) \qquad (5)$$
$$+ \left(h_2 + \frac{1}{2} k_2^2\right) \lambda_{\max}(P^{-1} Y_i^T Y_i P^{-1}).$$

**Theorem 1.** *Given* $\alpha > 0$. *System (3) is* $\alpha$*-exponentially stabilizable if there exist symmetric positive definite matrices P, Q, R, M, matrices* $Y_i$ *and numbers* $\beta_i \ge 0, i \in \overline{m}$, $\sum_{i=1}^{m} \beta_i > 0$, *such that the following LMIs hold:*

$$i) \ \sum_{i=1}^{m} \beta_i L_i < 0, \qquad (6)$$

$$ii)\begin{bmatrix} M & U_i & \mu Y_i^T \\ U_i^T & H & 0 \\ \mu Y_i & 0 & I \end{bmatrix} > 0, \quad i \in \overline{m}. \tag{7}$$

*The switching rule is chosen as* $\sigma(x(t)) = i$ *whenever* $x(t) \in \overline{s}_i$ *and the feedback controller is given by:*

$$u(t) = Y_\sigma P^{-1} x(t),\ t \geq 0. \tag{8}$$

*Moreover, the solution* $x(t,\phi)$ *of the closed-loop system satisfies:*

$$\| x(t,\phi) \| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\alpha t} \|\phi\|,\ t \geq 0.$$

**Proof**

It follows from (6), that the system matrices $\{L_i\}$ is strictly complete, so we have $\bigcup_{i=1}^{m} \Omega_i = \mathbb{R}^n \setminus \{0\}$. Based on the set $\Omega_i$ we construct the sets $\overline{s}_i$ and we will show that:

$$\bigcup_{i=1}^{m} \overline{s}_i = \mathbb{R}^n \setminus \{0\},\ \overline{s}_i \cap \overline{s}_j = \varnothing,\ i \neq j. \tag{9}$$

Obviously $\overline{s}_i \cap \overline{s}_j = \varnothing,\ i \neq j$. For any $x \in \mathbb{R}^n \setminus \{0\}$ there $i \in \overline{m}$ such that y= $P^{-1}x \in \Omega_i$. Then we have, $x$= Py $\in s_i$. Therefore $\bigcup_{i=1}^{m} s_i = \mathbb{R}^n \setminus \{0\}$ and by the construction of sets $\overline{s}_i$ it follows that $\bigcup_{i=1}^{m} \overline{s}_i = \mathbb{R}^n \setminus \{0\}$.

The switching rule is chosen as $\sigma(x(t)) = i$ whenever $x(t) \in \overline{s}_i$ (this switching rule is well defined due to (9)). So when $x(t) \in \overline{s}_i$, the *i*th subsystem is activated and then we have the following subsystem:

$$\dot{x}(t) = A_i x(t) + D_i x(t - h_1(t)) + E_i \int_{t-k_1(t)}^{t} x(s)ds + B_i u(t) + C_i u(t - h_2(t)) + F_i \int_{t-k_2(t)}^{t} u(s)ds. \tag{10}$$

Denote $X = P^{-1}$, $Q_1 = XQX$, $R_1 = XRX$ and consider the following Lyapunov–Krasovskii functional for the closed-loop system of (10), where $u(t) = Y_i P^{-1} x(t)$:

$$V(x_t) = \sum_{i=1}^{5} V_i(x_t), \tag{11}$$

where: $V_1(x_t) = x^T(t) X x(t),$

$$V_2(x_t) = \int_{t-h_1(t)}^{t} e^{2\alpha(s-t)} x^T(s) Q_1 x(s) ds,$$

$$V_3(x_t) = \int_{t-k_1(t)}^{t} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^T(\theta) R_1 x(\theta) d\theta ds,$$

$$V_4(x_t) = \int_{t-h_2(t)}^{t} e^{2\alpha(s-t)} \| u(s) \|^2 ds,$$

$$V_5(x_t) = \int_{t-k_2(t)}^{t} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \| u(t+\theta) \|^2 d\theta ds.$$

It is easy to verify that:

$$\alpha_1 \| x(t) \|^2 \le V(x_t) \le \alpha_2 \| x_t \|^2 \tag{12}$$

where $\alpha_1$, $\alpha_2$ is defined in (5).

Taking derivative of $V_1(x_t)$ with respect to *t* along the solution $x(t)$ of the system, we have:

$$\dot{V}_1(x_t) = x^T(t)\Big[A_i^T X + XA_i + X\left(B_i Y_i + Y_i^T B_i^T\right)X\Big]x(t) + 2x^T(t)XD_i x(t-h_1(t))$$

$$+2x^T(t)XC_i u(t-h_2(t)) + 2x^T(t)XE_i \int_{t-k_1(t)}^{t} x(s)ds + 2x^T(t)XF_i \int_{t-k_2(t)}^{t} u(s)ds\,. \tag{13}$$

Applying Lemma 1 and 2 gives:

$$2x^T(t)XE_i \int_{t-k_1(t)}^{t} x(s)ds \le k_1 e^{2\alpha k_1} x^T(t) XE_i R_1^{-1} E_i^T Xx(t) + e^{-2\alpha k_1} \int_{t-k_1}^{t} x^T(s) R_1 x(s) ds, \tag{14}$$

$$2x^T(t)XF_i \int_{t-k_2(t)}^{t} u(s)ds \le k_2 e^{2\alpha k_2} x^T(t) XF_i F_i^T Xx(t) + e^{-2\alpha k_2} \int_{t-k_2}^{t} \| u(s) \|^2 ds. \tag{15}$$

Therefore, from (13) to (15) we have:

$$\begin{aligned}\dot{V}_1(x_t) \le{}& x^T(t)\Big[A_i^T X + XA_i + X\left(B_i Y_i + Y_i^T B_i^T\right)X\Big]x(t)\\ &+2x^T(t)XD_i x(t-h_1(t)) + 2x^T(t)XC_i u(t-h_2(t))\\ &+k_1 e^{2\alpha k_1} x^T(t)XE_i R_1^{-1} E_i^T Xx(t) + e^{-2\alpha k_1} \int_{t-k_1}^{t} x^T(s) R_1 x(s) ds\\ &+k_2 e^{2\alpha k_2} x^T(t) XF_i F_i^T Xx(t) + e^{-2\alpha k_2} \int_{t-k_2}^{t} \| u(s) \|^2 ds.\end{aligned} \tag{16}$$

Next, taking derivative of $V_2(x_t)$, $V_3(x_t)$, $V_4(x_t)$, $V_5(x_t)$ with respect to *t* along the solution $x(t)$ of the system, we have:

$$\dot{V}_2(x_t) = x^T(t) Q_1 x(t) - (1 - \dot{h}_1(t)) + e^{-2\alpha h_1(t)} x^T(t-h_1(t)) Q_1 x(t-h_1(t)) - 2\alpha V_2$$

$$\leq x^T(t)Q_1x(t)-(1-\delta_1)e^{-2\alpha h_1}x^T(t-h_1(t))Q_1x(t-h_1(t))-2\alpha V_2\,,$$

$$\dot{V}_3(x_t)\leq k_1x^T(t)R_1x(t)-e^{-2\alpha k_1}\int\limits_{t-k_1}^{t}x^T(s)R_1x(s)ds-2\alpha V_3\,,$$

$$\dot{V}_4(x_t)=u^T(t)u(t)-(1-\dot{h}_2(t))e^{-2\alpha h_2(t)}\,\|u(t-h_2(t))\|^2-2\alpha V_4 \tag{17}$$

$$\leq x^T(t)XY_i^TY_iXx(t)-(1-\delta_2)e^{-2\alpha h_2}\,\|u(t-h_2(t))\|^2-2\alpha V_4\,,$$

$$\dot{V}_5(x_t)\leq k_2x^T(t)XY_i^TY_iXx(t)-\int\limits_{t-k_2}^{t}\|u(s)\|^2ds-2\alpha V_5\,.$$

Combining (16) and (17) implies:

$$\begin{aligned}\dot{V}(x_t)+2\alpha V(x_t)\leq{}& x^T(t)\left[A_i^TX+XA_i+X\left(B_iY_i+Y_i^TB_i^T\right)X\right]x(t)+2\alpha V_1\\
&+2x^T(t)XD_ix(t-h_1(t))+2x^T(t)XC_iu(t-h_2(t))\\
&+k_1e^{2\alpha k_1}x^T(t)XE_iR_1^{-1}E_i^TXx(t)+k_2e^{2\alpha k_2}x^T(t)XF_iF_i^TXx(t)\\
&+x^T(t)XQXx(t)+k_1^2x^T(t)XRXx(t)+(1+k_2^2)x^T(t)XY_i^TY_iXx(t)\\
&-(1-\delta_1)e^{-2\alpha h_1}x^T(t-h_1(t))Q_1x(t-h_1(t))-(1-\delta_2)e^{-2\alpha h_2}\,\|u(t-h_2(t))\|^2\,.\end{aligned}$$

Applying Lemma 1 and 2 gives:

$$\begin{aligned}&2x^T(t)XD_ix(t-h_1(t))-(1-\delta_1)e^{-2\alpha h_1}x^T(t-h_1(t))Q_1x(t-h_1(t))\\
&\leq\frac{1}{1-\delta_1}e^{2\alpha h_1}x^T(t)XD_iQ_1^{-1}D_i^TXx(t)\,,\end{aligned}$$

$$2x^T(t)XC_iu(t-h_2(t))-(1-\delta_2)e^{-2\alpha h_2}\,\|u(t-h_2(t))\|^2\leq\frac{1}{1-\delta_2}e^{2\alpha h_2}x^T(t)XC_iC_i^TXx(t).$$

Therefore, we obtain:

$$\begin{aligned}\dot{V}(x_t)+2\alpha V(x_t)\leq{}& x^T(t)\left[A_i^TX+XA_i+2\alpha X\right]x(t)\\
&+x^T(t)X[\mathrm{B}_i\mathrm{Y}_i+\mathrm{Y}_i^T\mathrm{B}_i^T+\mathrm{Q}+\mathrm{k}_1\mathrm{R}+(1+\mathrm{k}_2)\mathrm{Y}_i^T\mathrm{Y}_i\,]\mathrm{X}x(\mathrm{t})\\
&+\frac{1}{1-\delta_1}e^{2\alpha h_1}x^T(t)XD_iQ_1^{-1}D_i^TXx(t)+k_1e^{2\alpha k_1}x^T(t)XE_iR_1^{-1}E_i^TXx(t)\\
&+\frac{1}{1-\delta_2}e^{2\alpha h_2}x^T(t)XC_iC_i^TXx(t)+k_2e^{2\alpha k_2}x^T(t)XF_iF_i^TXx(t)\\
={}& x^T(t)\left[XPA_i^TX+XA_iXP+XP2\alpha X\right]x(t)\\
&+x^T(t)X\left[\mathrm{B}_i\mathrm{Y}_i+\mathrm{Y}_i^T\mathrm{B}_i^T+\mathrm{Q}+\mathrm{k}_1\mathrm{R}+\left(\frac{1}{1-\delta_2}e^{2\alpha h_2}C_iC_i^T+k_2e^{2\alpha k_2}F_iF_i^T\right)\right]Xx(t)\end{aligned}$$

$$+x^T(t)X\left[\frac{1}{1-\delta_1}e^{2\alpha h_1}D_iQ_1^{-1}D_i^T+k_1e^{2\alpha k_1}E_iR_1^{-1}E_i^T+(1+k_2)Y_i^TY_i\right]Xx(t)$$

**Since** $U_i=[D_iP \quad E_iP]$, $H=\operatorname{diag}\left[(1-\delta_1)e^{-2\alpha h_1}Q,\frac{1}{k_1}e^{-2\alpha k_1}R\right]$, we have:

$$UH^{-1}U^T=\left[\frac{1}{1-\delta_1}e^{2\alpha h_1}D_iPQ^{-1}PD_i^T+k_1e^{2\alpha k_1}E_iPR^{-1}PE_i^T\right]$$

$$=\left[\frac{1}{1-\delta_1}e^{2\alpha h_1}D_i(XQX)^{-1}D_i^T+k_1e^{2\alpha k_1}E_i(XRX)^{-1}E_i^T\right]$$

$$=\left[\frac{1}{1-\delta_1}e^{2\alpha h_1}D_iQ_1^{-1}D_i^T+k_1e^{2\alpha k_1}E_iR_1^{-1}E_i^T\right].$$

where $\eta(t)=Xx(t)$, $\Xi_i=U_iH^{-1}U_i^T+(1+k_2)Y_i^TY_i$.

Moreover, we have: $G_i=B_iY_i+Y_i^TB_i^T+\left(\frac{1}{1-\delta_2}e^{2\alpha h_2}C_iC_i^T+k_2e^{2\alpha k2}F_iF_i^T\right)$

Therefore: $\dot V(x_t)+2\alpha V(x_t)\le\eta^T(t)\left(PA_i^T+A_iP+2\alpha P+G_i+Q+k_1^2R\right)\eta(t)+\eta^T(t)\Xi_i\eta(t).$

By Schur complement theorem (Lemma 3), (7) implies that:

$$M>U_iH^{-1}U_i^T+(1+k_2)Y_i^TY_i,\ i=1,2,\ldots,m.$$

Therefore: $\dot V(x_t)+2\alpha V(x_t)\le\eta^T(t)L_i\eta(t),\ t\ge0$.

Noting that $x(t)\in\bar{s}_i$ implies $\eta(t)=Xx(t)\in\Omega_i$ and $\eta^T(t)L_i\eta(t)\le0$ (According to (6) and Remark 1, the system $\{L_i\}$ is strictly complete), we have:

$$\dot V(x_t)+2\alpha V(x_t)\le0,\ \forall t\ge0.$$

Hence: $\dot V(x_t)\le V(\phi)e^{-2\alpha t}\le\alpha_2\|\phi\|^2e^{-2\alpha t},\ t\ge0$.

On the other hand, we have: $\alpha_1\|x(t)\|^2\le V(x_t)$ therefore

$$\|x(t,\phi)\|\le\sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\alpha t}\|\phi\|,\ t\ge0.$$

where $\alpha_1$, $\alpha_2$ is defined in (5). The proof is complete. □

**Example.** Consider switching system (3):

$$\dot x(t)=A_ix(t)+D_ix(t-h_1(t))+E_i\int_{t-k_1(t)}^{t}x(s)ds+B_iu(t)+C_iu(t-h_2(t))+F_i\int_{t-k2(t)}^{t}u(s)ds,\ i=\{1,2\},\qquad(18)$$

where: $h_1(t)=\sin^2\frac{1}{2}t$, $h_2(t)=\frac{1}{2}\sin^2 t$, $k_1(t)=0.9\cos^2 t$, $k_2(t)=0.9\cos^2\frac{1}{3}t$,

$$h_1 = h_2 = 1,\ k_1 = k_2 = 0.9,\ \delta_1 = \delta_2 = 0.5$$

and $A_1 = \begin{bmatrix} -15 & 2 \\ -5 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, E_1 = \begin{bmatrix} 2 & 1 \\ 0 & -5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$

$$A_2 = \begin{bmatrix} 5 & -1 \\ 2 & -45 \end{bmatrix}, D_2 = \begin{bmatrix} 1 & -5 \\ 2 & -7 \end{bmatrix}, E_2 = \begin{bmatrix} -2 & 1 \\ -4 & 5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For $\alpha = 0.3$ and $\beta_1 = \beta_2 = 0.5$. By using LMIs in Matlab, we can check the system (18) is exponentially stabilizable with (The LMIs (6) and (7) in Theorem 1 are satisfied):

$$P = \begin{bmatrix} 5.8428 & -0.4086 \\ -0.4086 & 2.7547 \end{bmatrix},\ Q = \begin{bmatrix} 96.7740 & -29.5891 \\ -29.5891 & 204.9731 \end{bmatrix},$$

$$R = \begin{bmatrix} 22.0317 & -8.7827 \\ -8.7827 & 52.6665 \end{bmatrix},\ M = \begin{bmatrix} 28.0237 & 3.9155 \\ 3.9155 & 49.9604 \end{bmatrix},$$

$$Y_1 = [\text{-}0.8859\ \ \text{-}2.1944], \qquad Y_2 = [0.0707\ \ \text{-}0.1583].$$

Then the system of matrices $\{L_1, L_2\}$, where

$$L_1 = \begin{bmatrix} -125.7877 & -25.9570 \\ -25.9570 & 113.7972 \end{bmatrix},\ L_2 = \begin{bmatrix} 115.7917 & 19.5674 \\ 19.5674 & -149.3144 \end{bmatrix}.$$

are strictly complete. The switching regions are constructed by:

$$s_1 = \{(x_1, x_2) \in \mathbb{R}^2 \text{: -}6.0362x_1^2\text{-}1.8188x_1x_2 + 2.7007x_2^2 < 0\},$$
$$s_2 = \{(x_1, x_2) \in \mathbb{R}^2 \text{: } 5.5713x_1^2 + 1.5161x_1x_2\text{-}3.4942x_2^2 < 0\},$$

$$\overline{s}_1 = s_1,\ \overline{s}_2 = s_2 \setminus s_1.$$

With the switching rule $\sigma(x(t)) = i$ whenever $x(t) \in \overline{s}_i$, $i = \{1, 2\}$ and the state feedback controller $u(t) = K_\sigma x(t)$, $t \geq 0$, where:

$$K_1 = [\text{-}0.2095\ \ \text{-}0.8277],\ K_2 = [0.0082\ \ \text{-}0.0563],$$

the system (18) is 0.3-exponentially stabilizable.

Moreover, every solution of $x(t,\phi)$ the closed-loop system satisfies:

$$\| x(t,\phi) \| \leq 13.5851 e^{-0.3t} \| \phi \|,\ t \geq 0.$$

## 4. Conclusion

By using the Lyapunov function method in combination with matrix knowledge, we have given a sufficient condition for "Exponential stabilization of the class of the switched systems with mixed time varying delays in state and control". We also give a numerical example to illustrate this problem.

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