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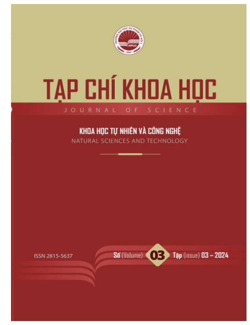

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A probabilistic method to prove AM-QM inequality

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Abstract

Inequality is one of the interesting topics in mathematics that attracts the attention of both students and researchers. Solving inequalities often requires creativity, making it a challenging topic for many students. The inequality of arithmetic and quadratic means, or AM-QM in short, states that the arithmetic mean of a list of non-negative real numbers is less than or equals to the square root of the quadratic mean of the same list. There are many methods of proving the AM-QM inequality. In this paper, we will present a probabilistic method to prove the weighted general AM-QM inequality and show that the classical AM-QM inequality is a special case of the generalized AM-QM inequality with equal weights.

Keywords: Inequalities, AM-QM inequality, probabilistic method, mean, discrete random variables

1. Introduction

The AM-QM inequality has many applications in different fields: finance, physics, computer science, etc. The AM-QM inequality states that the arithmetic mean of a list of non-negative real numbers is less than or equals to the square root of the quadratic mean of the same list, that is for any list of non-negative real numbers $a_1, a_2, ..., a_n$, the following inequality holds

$$\frac{a_1 + a_2 + ... + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}}.$$

The equality occurs if and only if $a_1 = a_2 = ... = a_n$. [1]–[8]

There are many methods of proving the AM-QM inequality such as the induction method, the method using the Cauchy-Schwarz inequality, the Lagrange multiplier method and the method using the Jensen

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inequality [1]–[8]. These proof methods contribute to the beauty of mathematics in general and inequalities in particular.

Besides that, the probability method was first proposed by Paul Erdos in [9]. Since then, this method has been widely used in combinatorial theory and random graphs [10]. The basic idea of the probability method is to prove the existence of a certain combinatorial structure, we construct a suitable probability space and show that a randomly selected element in this space has the expected property with a positive probability [11]–[19].

In this paper, we demonstrate a probabilistic method to prove the weighted general AM-QM inequality and show that the classical AM-QM inequality is a special case of the generalized AM-QM inequality with equal weights.

2. Probabilistic method to prove AM-QM inequality

2.1. Preliminaries

For the convenience of the readers, we restate some concepts and results related to discrete random variables with characteristic numbers as follows.

Let X be a discrete random variable defined on a probability space (Ω, F, P), where Ω is the sample space, F is the sigma algebra of subsets of Ω, and P the probability measure on F. In this paper, we only consider that discrete random variable X takes finitely values, in this case the set of all possible values of X is $X(\Omega)=\{x_1, x_2, ..., x_n\}$ and its probability function $p_i = P(X = x_i), i = 1,2,...,n.$ Note that $\sum_{i=1}^{n} p_i = 1.$

Definition 2.1. [20, Def. 4.1] The expected value of the discrete random variable X is the number denoted by $E[X]$ and defined as follows

$$E[X] = \sum_{i=1}^{n} x_i p_i.$$

From the definition, we have some basic properties of the expected value as follows.

Proposition 2.2. [20, Th. 4.4] The following statements hold:

1. For any real number a, $E[a] = a$.

2. If X is a nonnegative discrete random variable, then $E[X] \geq 0$.

3. For all discrete random variable X, and for any real numbers a, b,

$$E[aX \pm b] = aE[X] \pm b.$$

4. Let X and Y be discrete random variables and let a, b be real numbers. Then,

$$E[aX \pm bY] = aE[X] \pm bE[Y].$$

Definition 2.3. [20, Def.4.2] The variance of the discrete random variable X is the nonnegative number denoted by $Var[X]$ and defined as follows

$$Var[X] = E\left[X - E[X]\right]^2.$$

Note that from the definition of the variance and properties of the expectation, we have

$$\begin{aligned}Var[X] &= E\left[X - E[X]\right]^2 \\ &= E\left[X^2 - 2XE[X] + (E[X])^2\right] \\ &= E[X^2] - (E[X])^2.\end{aligned}$$

Thus, the variance of the discrete random variable X can be calculated as follows.

$$\begin{aligned}Var[X] &= E[X^2] - (E[X])^2 \\ &= \sum_{i=1}^{n} x_i^2 p_i - \left(\sum_{i=1}^{n} x_i p_i\right)^2.\end{aligned}$$

Some basic properties of the variance are given in the following proposition.

Proposition 2.4. [20, Th.4.9] *The following statements hold:*

1. For any real number a, $Var[a] = 0$.

2. For any discrete random variable X, *and for any real numbers* a, b,

$$Var[aX \pm b] = a^2 Var[X].$$

3. For any discrete random variable X,

$$E[X^2] \geq \left(E[X]\right)^2.$$

Next, we recall the Cauchy-Schwarz inequality for random variables as follows.

Proposition 2.5. [20, Th.4.14, Cauchy-Schwarz Inequality] Let X and Y be random variables such that $E[X^2]$ and $E[Y^2]$ exist. Then

$$\left(E[XY]\right)^2 \leq E[X^2]E[Y^2].$$

2.2. Main results

The main objective of this section is to present the probability method of proving the weighted AM-QM inequality as follows.

Theorem 2.6. *For any list of n real numbers* $a_1, a_2, ..., a_n$ *and any list of n nonnegative real numbers* $b_1, b_2, ..., b_n$ *such that* $\sum_{i=1}^{n} b_i = 1$. *Then, the following inequality holds*

$$\sum_{i=1}^{n} a_i^2 b_i \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2. \quad (1)$$

Proof. Let $A = \sum_{i=1}^{n} a_i b_i$ and construct the discrete random variable X that have the probability distribution defined as follows

$$p_X(x) = P(X = x) = \begin{cases} b_i & \text{if } x = \dfrac{a_i}{A}, i = 1, 2, ..., n \\ 0 & \text{otherwise} \end{cases}$$

From the definition of X, we get

$$E[X] = \sum_{i=1}^{n} \frac{a_i b_i}{A}$$

and

$$E[X^2]=\sum_{i=1}^{n}\frac{a_i^2 b_i}{A^2}.$$

Using property 3. of Proposition 2.4, we have

$$\sum_{i=1}^{n}\frac{a_i^2 b_i}{A^2}\geq\left(\sum_{i=1}^{n}\frac{a_i b_i}{A}\right)^2$$

or

$$\sum_{i=1}^{n}a_i^2 b_i\geq\left(\sum_{i=1}^{n}a_i b_i\right)^2.$$

Obviously, the equality holds if and only if $a_1=a_2=...=a_n$ and $b_1=b_2=...=b_n$. Therefore, the weighted AM-QM inequality is proven. □

Corollary 2.7. [Classical AM-QM inequality] *For any list of n real numbers* $a_1,a_2,...,a_n$, *the following inequality holds*

$$\frac{a_1+a_2+....+a_n}{n}\leq\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}}. \quad (2)$$

Proof. Applying Theorem 2.3. with equal weights, that is, $b_i=\frac{1}{n}, i=1,2,...,n,$ we obtain the classical AM-QM inequality. □

Remark 2.8. In some other proof methods such as the method using the Cauchy-Schwarz inequality, for example, in order to use the Cauchy-Schwarz inequality, real numbers have to be positive. However, in the probability method in the above proof, the real numbers $a_1,a_2,...,a_n$ are not required to be positive. Therefore, inequality (2) is more general than the classical AM-QM inequality.

Remark 2.9. The probability method can be used to prove several classes of inequalities. However, for each class, it is important to build appropriate random variables and then apply the properties of expectation and moment to achieve the desired inequality.

Next, we present some ways to construct discrete random variables to prove inequalities.

Example 2.10. [cf. 6] Let a,b,c be positive real numbers with $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}. \quad (3)$$

Proof. In [6] the authors Fuhua Wei and Shanhe Wu presented 10 ways to prove inequality (3), of which the 5th way uses the probability method to prove it. We restate the proof as follows. First, let $s=ab+ac+bc$ and

$$A=\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}.$$

Then, from the assumption $abc=1$, we have

$$\begin{aligned} A &= \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \\ &= \frac{b^2c^2}{ab+ac} + \frac{a^2c^2}{bc+ba} + \frac{a^2b^2}{ca+cb} \\ &= bc\frac{bc}{ab+ac} + ac\frac{ac}{bc+ba} + ab\frac{ab}{ca+cb}. \end{aligned}$$

Next, we construct the discrete random variable X with the following probability distribution as follows

$$p_X(x) = P(X=x) = \begin{cases} \dfrac{ab+ac}{2s} & \text{if } x = \dfrac{bc}{ab+ac} \\ \dfrac{bc+ba}{2s} & \text{if } x = \dfrac{ac}{bc+ba} \\ \dfrac{ca+cb}{2s} & \text{if } x = \dfrac{ab}{ca+cb} \\ 0 & \text{otherwise} \end{cases}$$

Thus, we get

$$E[X] = \frac{bc}{2s} + \frac{ac}{2s} + \frac{ab}{2s} = \frac{1}{2}$$

and

$$E[X^2] = \frac{b^2c^2}{2s(ab+ac)} + \frac{a^2c^2}{2s(bc+ba)} + \frac{a^2b^2}{2s(ca+cb)}.$$

From the property of moment $E[X^2] \ge \left(E[X]\right)^2$, we obtain

$$\frac{b^2c^2}{(ab+ac)} + \frac{a^2c^2}{(bc+ba)} + \frac{a^2b^2}{(ca+cb)} \ge \frac{s}{2} \overset{AM-GM}{\ge} \frac{3}{2}\sqrt[3]{a^2b^2c^2} = \frac{3}{2}.$$

The equality occurs if and only if $a=b=c$. Therefore, the proof is complete. □

Example 2.11. [cf. 3] Let a,b,c be positive real numbers with $abc=1$. Prove that

$$\frac{a^2}{(b+c)} + \frac{b^2}{(c+a)} + \frac{c^2}{(a+b)} \ge \frac{3}{2}. \tag{4}$$

Proof. First, we let $s=a+b+c$ and construct the discrete random variable X with the following probability distribution as follows

$$p_X(x) = P(X=x) = \begin{cases} \dfrac{b+c}{2s} & \text{if } x = \dfrac{a}{b+c} \\ \dfrac{c+a}{2s} & \text{if } x = \dfrac{b}{c+a} \\ \dfrac{a+b}{2s} & \text{if } x = \dfrac{c}{a+b} \\ 0 & \text{otherwise} \end{cases}$$

Hence, we get

$$E[X] = \frac{a}{2s} + \frac{b}{2s} + \frac{c}{2s} = \frac{1}{2}$$

and

$$E[X^2] = \frac{a^2}{2s(b+c)} + \frac{b^2}{2s(c+a)} + \frac{c^2}{2s(a+b)}.$$

Arguing similarly to Example 2.9, we obtain

$$\frac{a^2}{(b+c)} + \frac{b^2}{(c+a)} + \frac{c^2}{(a+b)} \geq \frac{s}{2} \overset{AM-GM}{\geq} \frac{3}{2}\sqrt[3]{abc} = \frac{3}{2}.$$

The equality occurs if and only if $a = b = c$. Therefore, the proof is complete. □

Next, we present the way by using the inequality in Proposition 2.5. to prove Cauchy-Bunyakovsky-Schwarz inequality for real numbers as follows.

Example 2.12. [Cauchy-Bunyakovsky-Schwarz inequality, cf. 3] For any lists of real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{5}$$

Proof. We construct the discrete random vector (X, Y) with the joint probability distribution as follows

$$p_{X,Y}(x,y) = P(X = x, Y = y) = \begin{cases} \frac{1}{n} & \text{if } (x,y) = (a_i, b_i), i = 1,2,...,n \\ 0 & \text{otherwise} \end{cases}$$

Then, we get the probability distributions of X and Y as follows

$$p_X(x) = P(X = x) = \begin{cases} \frac{1}{n} & \text{if } x = a_i, i = 1,2,...,n \\ 0 & \text{otherwise} \end{cases}$$

and

$$p_Y(y) = P(Y = y) = \begin{cases} \frac{1}{n} & \text{if y} = b_i, i = 1,2,...,n \\ 0 & \text{otherwise} \end{cases}$$

Thus, we obtain

$$E[XY] = \frac{1}{n}\sum_{i=1}^{n} a_i b_i, E[X^2] = \frac{1}{n}\sum_{i=1}^{n} a_i^2, E[Y^2] = \frac{1}{n}\sum_{i=1}^{n} b_i^2.$$

Applying Proposition 2.5., we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i^2\right)$$

or equivalently

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Obviously, the equality occurs if and only if $a_i = b_i, i = 1,2,...,n.$ Therefore, the proof is complete. □

3. Conclusion

The AM-QM inequality has various proof methods. In this article, we have introduced the proof method using techniques and results in probability theory. Specifically, we have constructed a suitable discrete random variable and then use the property of moments to obtain the weighted general AM-QM inequality. This method can be applied to prove several other classes of inequalities.

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