HPU2. Nat. Sci. Tech. Vol 1, issue 01 (2022), 40-45

HPU2 Journal of Sciences:
Natural Sciences and Technology

journal homepage: https://sj.hpu2.edu.vn

Article type: Research article

Weak* fixed point property of fourier-stieltjes algebra on compact groups

Ngoc-Muoi Bui[a,*], Thi-Thu Tran[a], Quoc-Tuan Nguyen[a]

[a]*Hanoi Pedagogical University 2, 32 Nguyen Van Linh, Phuc Yen, Vinh Phuc, Vietnam*

Abstract

Lau and Mah [3] showed that a Fourier-Stieltjes algebra $B(G)$ on a separable compact group G has the weak* fixed point property, i.e. every nonexpansive mapping on a weak* compact convex subset of $B(G)$ has a fixed point. We extend this result by showing that a similar fixed point property holds for norm continuous and asymptotically nonexpansive mappings.

Keywords: Fourier-Stieltjes algebra, semitopological semigroup, fixed point property.

1. Introduction

Let K be a non-empty convex subset of a Banach space X. Let $T: K \to K$ be a *nonexpansive* map, namely $\|Tx - Ty\| \le \|x - y\|$, for all x, y in K. Schauder [5] shows that T has a fixed point if K is norm compact. However, if K is weakly (resp. weak*) compact convex subset of a Banach (resp. dual Banach) space, then T does not necessarily have a fixed point, see [1] for more discussion. We called a dual Banach space X has *weak* fixed point property* if every nonexpansive mapping on a weak* compact convex subset of X has a fixed point.

Let $L^1(G)$ be the group algebra of G associated with the regular left Haar measure $d\lambda$ with convolution product. Define a norm on $L^1(G)$ by

$$\|f\|_* = \sup \|\pi(f)\|,$$

where the supremum is taken over all nondegenerate $* -$ representations $\pi: L^1(G) \to B(H_\pi)$ for some Hilbert space H_π. Let $C^*(G)$ be the completion of $L^1(G)$ with respect to the $\|.\|_*$ norm.

Let $P(G)$ be the set of all positive definite functions on G, i.e. for each function $f \in P(G)$ there is a unitary representation $\pi: G \to B(H_\pi)$ for some Hilbert space H_π and some $\xi \in H_\pi$ such that

---

*Corresponding author. E-mail: buingocmuoi@hpu2.edu.vn

https://doi.org/10.56764/hpu2.jos.2022.1.1.54-61

Received date: 23-8-2022 ; Revised date: 26-8-2022 ; Accepted date: 30-8-2022

$f(s)=\langle \pi(s)\xi,\xi\rangle$ for all $s\in G$. Let $B(G)$ be the linear span of $P(G)$. In this way, $B(G)$ becomes a Banach algebra with the pointwise multiplication, and it is the dual space of $C^*(G)$ with the norm defined by

$$\|\varphi\|_{B(G)}=\sup\left\{\left|\int \varphi(t)f(t)d\lambda(t)\right|: f\in L^1(G),\|f\|_*\le 1\right\}.$$

A *semitopological semigroup* S is a semigroup with a Hausdorff topology such that the product is separately continuous, i.e., for each fixed $t\in S$, both the maps $s\mapsto ts$ and $s\mapsto ts$ from S into S are continuous. We call S *left reversible* (resp. *right reversible*) if any two right ideals (resp. left ideals) of S always intersect, in other words, for each $s,t\in S$ we have $\overline{sS}\cap\overline{tS}\neq\varnothing$ (resp. $\overline{Ss}\cap\overline{St}\neq\varnothing$). We call S *reversible* if it is both left and right reversible. Examples of reversible semigroup includes topological groups, commutative semitopological semigroups and discrete inverse semigroups.

An *action of* a semitopological semigroup S on a Hausdorff topological space K is a mapping of $S\times K$ into K, denoted by $(s,x)\mapsto s.x$, such that $(st)x=s.(tx)$ for all $s,t\in S$ and $x\in K$. We call the action *continuous* if the mapping $(s,x)\mapsto T_s x$ is separately continuous. A point $x_0\in K$ is called a *common fixed point* for S if $s.x_0=x_0$ for all $s\in S$.

Definition 1.1 ([4]). An action $S\times K\mapsto K$ is called of

(i) asymptotically nonexpansive type if for each $x,y\in K$ and $\varepsilon>0$, there exist a left ideal I and a right ideal J in S such that

$$\|sx-sty\|\le\|s-ty\|+\varepsilon \quad \text{for all } s\in I \text{ and } t\in J; \tag{1.1}$$

(ii) strongly asymptotically nonexpansive type if for each $x,y\in K$, $\varepsilon>0$ and each right ideal J of S, there exist a left ideal I of S such that

$$\|sx-sty\|\le\|s-ty\|+\varepsilon \quad \text{for all } s\in I \text{ and } t\in J.$$

A map $T:K\mapsto K$ is called *asymptotically nonexpansive* if for each $x,y\in K$,

$$\lim_k \sup=\|T^kx-T^ky\|\le\|x-y\| \tag{1.2}$$

then is of asymptotically nonexpansive type. We have asymptotic nonexpansiveness is strictly weaker than nonexpansiveness, see [4] for an example. Furthermore, if a map T is asymptotically nonexpansive on K then the action of $S=\{T^k:k\in N\}$ on K is asymptotically nonexpansive type.

Lau and Mah [3] showed that

Theorem 1.2 ([3, Theorem 4.2]). *Let G be a separable compact group. Let K be a non-empty weak* compact convex subset of the Fourier-Stieltjes algebra $B(G)$. Let S be a left reversible semitopological semigroup. Let $S\times K\mapsto K$ be a nonexpansive, norm continuous action of S on K. Then S has a common fixed point in K.*

As a consequence, B(G) has a weak* fixed point property. Indeed, for a nonexpansive mapping T, consider the semigroup S = $\{T^k:k\in N\}$ generated by T. Then S has a common fixed point by Theorem

1.2, hence T has a fixed point. Motivated by Theorem 1.2, we show in this paper fixed point properties for a semigroup of asymptotically nonexpansive mappings. As a consequence B(G) has weak* fixed point property for asymptotically nonexpansive mappings. This provides a variance and generalization of some results in [3].

2. Materials and methods or Experiments

Let K be a nonempty subset of a Banach space X and let $\{D_\lambda : \lambda \in \Delta\}$ be a decreasing net of bounded non-empty subsets of X. For each x $\in$K and $\lambda \in \Delta$, let

$$r_\lambda(x) = \sup\{\|x-y\| : y \in D_\lambda\},$$
$$r(x) = \lim_\lambda r_\lambda(x) = \inf_\lambda r_\lambda(x), \tag{2.3}$$
$$r = \inf\{r(x) : x \in K\}.$$

The set (possibly empty)

$$AC(\{D_\lambda : \lambda \in \Delta\}) = \{x \in K : r(x) = r\}$$

Consists of *asymptotic centers*, and *r* is called the *asymptotic radius*, of $\{D_\lambda : \lambda \in \Delta\}$ with respect to K.

The following lemma arises from the proof of of [2, Theorem 3.1].

Lemma 2.1. *Let S be a right reversible semitopological semigroup. Assume $S \times K \to K$ is a separately continuous action of S on a compact convex subset K of a locally convex space. Then there exists a subset L_0 of K which is minimal with respect to being nonempty, compact, convex and satisfying the following conditions **(*1)** and **(*2).***

***(*1)** there exists a collection $\Delta = \{\Delta_i : i \in I\}$ of closed subsets of K such that $L_0 = \bigcap \Delta$, and*

***(*2)** for each $x \in L_0$ there is a left ideal $J_i \subseteq S$ such that $J_i.x \subseteq \Delta_i$ for each $i \in I$.*

Furthermore, L_0 contains a subset Y that is minimal with respect to being nonempty, compact, and S-invariant, $i.e., s.Y \subseteq Y$ for all $s \in S$.

Proof. We sketch the arguments in the proof of [2, Theorem 3.1]. By the Zorn's lemma such L_0 always exists. For each $x \in L_0$, let Φ be the collection of all finite intersections of sets in $\{\Delta_i : i \in I\}$. Order Φ by the reverse set inclusion. For any $\alpha = \Delta_1 \bigcap \Delta_2 \bigcap ... \bigcap \Delta_n \in \Phi$, choose left ideals J_i such that $J_i.x \subseteq \Delta_i$ for $i = 1,....,n$. By the right reversibility, there exists $s_\alpha \in \bigcap_{i=1}^n \overline{J_i}$. Thus, S $s_\alpha.x \subseteq \alpha$. Let z be a cluster point of the net $\{s_\alpha.x\}_{\alpha\in\Phi}$.

Then $\overline{Sz}$ is a closed S-invariant subset of L_0. By Zorn's lemma, there exists a minimal subset $Y \subseteq \overline{Sz} \subseteq L_0$ with respect to being nonempty, closed and S-invariant.

The following lemma is crucial for our results.

Lemma 2.2 ([3, Theorem 4.1]). *Let G be a separable compact group. Let K be a nonempty weak* closed convex subset of B(G). Let $\{D_\lambda : \lambda \in \Delta\}$ be a downward directed net of nonempty bounded subsets of K. Then the set of asymptotic centers of $\{D_\lambda : \lambda \in \Delta\}$ with respect to K is nonempty, norm-compact and convex.*

We establish in the following weak* fixed point properties of B(G) that provides a variance of [3] for various nonexpansive mappings.

Theorem 2.3 *Let G be a separable compact group. Let K be a non-empty weak* compact convex subset of A(G). Let* $S \times K \to K$ *be a norm continuous action of a semitopological semigroup S on K. Assume either*

(i) S is commutative and the action is of asymptotically nonexpansive type, or

(ii) S is reversible and the action is of strongly asymptotically nonexpansive type.

Then S has a common fixed point in K.

Proof. We follow the idea in proving Theorem 1.2 in [3]. For a fixed $z \in k$ and $s \in S$, let $W_s = \overline{sSz}$. Direct S by the order $s \geq t$ if $sS \subset \overline{tS}$. Then $\{W_s : s \in S\}$ is a decreasing net of subsets of K. By Lemma 2.2, the set C of asymptotic center of $\{W_s : s \in S\}$ with respect to K is a non-empty convex norm-compact set.

We show that for each x in C, there is a left ideal I of S such that $Ix \subset C$. Indeed, for each $\varepsilon > 0$, there is a $t \in S$ such that $r_t(x) = \sup\{\|x - y\| : y \in W_t\} \leq r + \frac{\varepsilon}{2}$. Hence $tSz \subset W_t \subset \overline{B}[x, r + \frac{\varepsilon}{2}]$, where $\overline{B}[x, r]$ is the closed ball center at x with radius r. Then

$$\|ts'.z - x\| \leq r + \frac{\varepsilon}{2}, \text{ for all } s' \in S. \quad (2.4)$$

Since the action is asymptotically nonexpansive type, for $x, t.z \in K$ and $\varepsilon > 0$, there are left ideals I and J of S such that

$$\|ss't.z - s.x\| \leq \leq \|s't.z - x\| + \frac{\varepsilon}{2} \leq r + \varepsilon, \text{ for all } s \in I \text{ and } s' \in J,$$

where the second inequality follow from (2.4) and the commutativity of S. Therefore, there exists an $s_1 \in tIJ \subset S$ such that

$$s_1Sz \subset IJtz \subset \overline{B}[s.x, r + \varepsilon].$$

Thus, $W_{s_1} \subset \overline{B}[s.x, r + \varepsilon]$. In other word, $sx \in C$ for all $s \in I$. Hence, $Ix \subset C$.

Similar to the case (ii), for given $x, z \in C, \varepsilon > 0$ and a right ideal tS of S, there is a left ideal I of S such that

$$\|s.x - stx'.z\| \leq \|x - ts'.z\| + \frac{\varepsilon}{2} \leq r + \varepsilon, \quad \text{for all } s \in I, s' \in S.$$

Hence $stSz \subset W_{st} \subset \overline{B}\ s.x, r + \varepsilon$, and thus $s.x \in C$ for all $s \in I$.

Follow Lemma 2.1 and the preceding discussion, there exists a subset L_0 of C which is minimal with respect to being nonempty, norm-compact, convex and satisfying the following conditions.

$*1$ there exists a collection $\Lambda = \ \Lambda_i : i \in I\ $ of closed subsets of C such that $L_0 = \cap\Lambda$, and

$(*2)$ for each $x \in L_0$ there is a left ideal $J_i \subseteq S$ such that $J_i.x \subseteq \Lambda_i$; for each $i \in I$. Furthermore, L_0 contains an S-invariant subset $\overline{Su}$ for some $u \in L_0$.

If L_0 contains one point then u is a common fixed point of S. Suppose that L_0 contains more than one point. For each $s \in S$, let $U_s = \overline{sSu}$. Then $\{U_s : s \in S\}$ is a decreasing net of subsets of L_0. Then the asymptotic center $AC(\{U_s : s \in S\})$ with respect to L_0 is a nonempty compact convex proper subset of Lo. Following an approach in [4], we show that $AC(\{U_s : s \in S\})$ also satisfies properties $(*1)$ and $(*2)$.

Let r be the asymptotic radius of $\{U_s : s \in S\}$ with respect to L_0. For each $n \in N$, let $C_n = \left\{x \in L_0 : r(x) \le r + \frac{1}{n}\right\}$. Then C_n is a nonempty closed convex subset of L_0. Moreover

$$AC(\{U_s : s \in S\}) = \bigcap_{n=1}^{\infty} C_n.$$

Let $x \in AC(\{U_s : s \in S\})$ and consider a fixed C_n. Since $x \in C_{3n}$, we have $r(x) \le r + \frac{1}{3n}$ Hence there exists an $t \in S$ such that

$$r_t(x) = \sup\{\|x - y\| : y \in U_t\} \le r + \frac{1}{2n}.$$

From (1.1), there are a left ideal I and a right ideal J of S such that

$$\|sx - ss'u\| \le \|x - s'u\| + \frac{1}{2n}$$

for all $s \in I$ and $s' \in J$. Take $t_0 \in \overline{J} \cap \overline{tS}$, we can assume $J = t_0 S$ for some $t_0 \in S$ such that $\overline{Ju} = U_{t_0} \subset U_t$ for all $t \in J$. Hence

$$\|sx - sy\| \le \|x - y\| + \frac{1}{2n} \le r + \frac{1}{n}$$

for all $s \in I$ and $y \in U_{t_0}$. Thus

$$\|sx - z\| \le r + \frac{1}{n}$$

for all $s \in I$ and $z \in U_{st_0} = \overline{st_0 Su}$. Hence

$$r(sx) \le r_{st_0}(sx) = \sup\{\|sx - y\| : y \in U_{st_0}\} \le r + \frac{1}{n}$$

for all $s \in I$. In other words, $Ix \subset C_n$. This implies that the proper subset $AC(\{U_s : s \in S\})$ of L_0 has also properties $(*1)$ and $(*2)$. The contradiction shows that L_0 consists one point and it is a common fixed point of S .

As consequence, $B(G)$ has the weak* fixed point property for asymptotically nonexpansive mappings.

Corollary 2.4. *Let G be a separable compact group, let K be a weak* compact convex subset of B(G). Let T be a norm-continuous and asymptotically nonexpansive map from K into K. Then T has a fixed point in K.*

Proof. Let $S = \{T^n : n \in N\}$ be the commutative discrete semigroup generated by T .Then the action of S on K is separately norm continuous and asymptotically nonexpansive type. From Theorem 2.3, S has a common fixed point in K , then so is T .

3. Conclusions

We have showed that if G is a separable compact group then the Fourier- Stieltjes algebra $B(G)$ has a fixed point property for the semigroup of asymptotically nonexpansive type mappings. As a consequence, $B(G)$ has the weak* fixed point property for asymptotically nonexpansive mappings. This extends some results of Lau and Mah [3] in literature.

Acknowledgments

This research was also funded by Hanoi Pedagogical University 2 Foudation for Sciences and Technology Development via grant numver HPU2.UT-2021.03

References

[1] D. Alspach, A fixed point free nonexpansive map, *Proc. Amer. Math. Soc.* (1981), 423-424.

[2] R. D. Holmes and A. T.-M. Lau, Asymptotically nonexpansive actions of topological semigroups and fixed points, *Bull. London Math. Soc*., 3 (1971), 343–347.

[3] A. T.-M. Lau and P. Mah, Fixed point properties for Banach algebras associated to locally compact groups, *J. Funct. Anal*., 258 (2010), 357–372.

[4] S. Saeidi, F. Golkar and A. M. Forouzanfar, Existence of fixed points for asymptotically nonexpansive type actions of semigroups, J. *Fired Point Theory Appl*., 20 (2018), no. 2, Art. 72.

[5] J. Schauder, Der Fixpunktsatz in Funktionalraumen, *Studia Math*., 2 (1930), 171-180.