HPU2. Nat. Sci. Tech. Vol 1, issue 01 (2022), 10-15

HPU2 Journal of Sciences:
Natural Sciences and Technology

journal homepage: https://sj.hpu2.edu.vn

Article type: Review article

A remark on global holderian error bounds for differentiable semi-algebraic functions

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Abstract

In this paper, we extend some results in [5] of H. V. Hà and P. D. Hoàng on global Holderian error bounds of the sub-level set:

$$\left[f \leq t\right] := \; x \in \mathbb{R}^n \mid f(x) \leq t$$

from polynomial functions to differentiable semi-algebraic functions. Moreover, we give some examples which show the difference between Fedoryuk set of polynomial functions and Fedoryuk set of differentiable semi-algebraic functions.

Keywords: Global Holderian error bounds, semi-algebraic functions, the Fedoryuk values.

1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable semi-algebraic function. For $t \in \mathbb{R}$, put $\left[f \leq t\right] := \; x \in \mathbb{R}^n \mid f(x) \leq t$ and $[a]_+ = \max\{a, 0\}$.

Definition 1.1 ([4]). We say that the non-empty set $\left[f \leq t\right]$ has a global Holderian error bound (GHEB for short) if there exist real numbers $\alpha, \beta, c > 0$ such that

$$\left[f(x) - t\right]_+^{\alpha} + \left[f(x) - t\right]_+^{\beta} \geq c\,\text{dist}\; x, \left[f \leq t\right] \text{, for all } x \in \mathbb{R}^n \qquad (1).$$

The existence of local and global error bounds has many applications to other problems, for instance, convergence analysis of algorithms, variational inequalities, sensitive analysis…. Research on error bounds has been attractive many authors, hence there have been many published articles in recent years (for example, see [3,7,9]). Some authors investigate stability of error bounds under the perturbations of value *t*, the results of the study stability have applications to research stability and

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https://doi.org/10.56764/hpu2.jos.2022.1.1.13-20

Received date: 24-8-2022 ; Revised date: 26-8-2022 ; Accepted date: 30-8-2022

convergence of iterative algorithms [10]. In this paper, we consider semi-algebraic functions and extend some results of the work [5]. More explicit, we want to answer the following question: Which values *t* such that $[f \le t]$ has a GHEB? In the other hand, how do we describe formula of the set $\Lambda_+(f) = \{t \in \mathbb{R} \mid [f \le t] \text{ has a GHEB}\}$? When is this set non-empty in the case of f is a continuous or differentiable semi-algebraic function? We relate the set $\Lambda_+(f)$ to special values of the so-called Fedoryuk values

$$\tilde{K}_\infty(f) := \{t \in \mathbb{R} : \exists \{x^k\} \subset \mathbb{R}^n, \|x^k\| \to \infty, \|\nabla f(x^k)\| \to 0, f(x^k) \to t\}.$$

The set of Fedoryuk values is a generalization of the set of critical values (see [6,8]). Using some examples, we point out that, in the case of differentiable semi-algebraic functions in two variables, results on semi-algebraic functions are different from results on polynomial functions. So, Theorem 3.6 and 3.7 in this paper are non-trivial extensions of the results in paper [5].

2. Materials and methods or Experiments

2.1. Some results in Semi-algebraic Geometry

In this section, we recall some notions and basic results in Semi-algebraic Geometry. These can be found in [1,2,6].

Definition 2.1. A semi-algebraic subset of $\mathbb{R}^n$ is the subset of $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients. In other words, the semi-algebraic subsets of $\mathbb{R}^n$ form the smallest class SA_n of subsets of $\mathbb{R}^n$ such that

i. If $P \in \mathbb{R}[x_1, \ldots, x_n]$, then $\{x \in \mathbb{R}^n \mid P(x) = 0\} \in SA_n$ and $\{x \in \mathbb{R}^n \mid P(x) > 0\} \in SA_n$;

ii. If $A \in SA_n$ and $B \in SA_n$, then $A \cup B, A \cap B$ and $\mathbb{R}^n \setminus A$ are in SA_n.

A semi-algebraic subset has the following structure.

Proposition 2.2 ([2]). *Every semi-algebraic subset of $\mathbb{R}^n$ is the union of finitely many semi-algebraic subsets of the form*

$$\{x \in \mathbb{R}^n \mid P(x) = 0, Q_1(x) > 0, \ldots, Q_l(x) > 0\}, \text{ where } l \in \mathbb{N} \text{ and } P, Q_1, \ldots, Q_l \in \mathbb{R}[x_1, \ldots, x_n].$$

The following theorem plays the important role in proofs of results in Semi-algebraic Geometry.

Theorem 2.3 (Tarski-Seidenberg [2]). *Let A be a semi-algebraic subset of $\mathbb{R}^{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, the projection on the first n coordinates. Then $\pi(A)$ is a semi-algebraic subset of $\mathbb{R}^n$.*

Definition 2.4 ([2]). Let $A \subset \mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is called semi-algebraic if its graph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y = f(x)\}$ is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}$.

We list some properties of semi-algebraic subsets and functions.

1. The unions, intersections and the complement of semi-algebraic sets are semi-algebraic sets. The Cartesian product of semi-algebraic sets is semi-algebraic. The closure, the interior and the boundary of a semi-algebraic set are semi-algebraic.

2. The composition of two semialgebraic mappings is semialgebraic. The direct image and the inverse image of a semialgebraic set by a semialgebraic function are semialgebraic.

3. If S is a semi-algebraic set, then distance function is defined by

$\operatorname{dist}(.,S): \mathbb{R}^n \to \mathbb{R}, x \mapsto \operatorname{dist}(x,S) := \inf\{\|x-y\| : y \in S\}$ is also semi-algebraic.

2.2. The existence of Holderian error bounds

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable semi-algebraic function and $t \in [\inf f, +\infty)$.

Definition 3.1 ([4]). Let S be a subset of $\mathbb{R}^n$. We say that

i) A sequence $\{x^k\} \subset \mathbb{R}^n$ is said to be a sequence of the first type of the set $[f \le t]$ on S if

$$\{x^k\} \subset S, \|x^k\| \to \infty, f(x^k) > t, f(x^k) \to t,$$
$$\exists \delta > 0 : \operatorname{dist}(x^k, [f \le t]) \ge \delta > 0.$$

ii) A sequence $\{x^k\} \subset \mathbb{R}^n$ is said to be a sequence of the second type of the set $[f \le t]$ on S if

$$\{x^k\} \subset S, \|x^k\| \to \infty, \exists M \in \mathbb{R}^n : t < f(x^k) \le M < +\infty,$$
$$\operatorname{dist}(x^k, [f \le t]) \to +\infty.$$

If $S = \mathbb{R}^n$, then we call sequences of the first and the second types.

The following gives us the necessary and sufficient conditions for the existence of the global Holderian error bound, it is the results in [3,7] and it is also the extension of Theorem A in [4]. The following characterization of global Holderian error bounds is based on two kinds of sequences.

Theorem 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous semi-algebraic function. Then, the following statements are equivalent:*

i) *There are no sequences of the first and second types $[f \le t]$;*

ii) $[f \le t]$ *has a GHEB, i.e., there exist $\alpha, \beta, c > 0$ such that*

$[f(x) - t]_+^\alpha + [f(x) - t]_+^\beta \ge c \operatorname{dist}(x, [f \le t])$, *for all* $x \in \mathbb{R}^n$.

Remark 3.3. We can extend above theorem for definable functions in some o-minimal structures (see [7]), an object of Real Algebraic Geometry and Real Analytic Geometry, which contains structure of semi-algebraic sets.

Put

$F_+^1 = \{t \in \mathbb{R} : [f \le t]$ has a sequence of the first type$\}$ and

$F_+^2 = \{t \in \mathbb{R} : [f \le t]$ has a sequence of the second type$\}$.

The set F_+^2 has an important property, this one is in the following proposition.

Proposition 3.4. *If $t \in F_+^2$ and $t' \le t$, then $t' \in F_+^2$.*

Proof. From the assumption, there exists a sequence of the second type of $[f \le t]$. Suppose that the sequence is $\{x^k\}$, then $\|x^k\| \to +\infty, \exists M \in \mathbb{R} : t < f(x^k) \le M$ such that $\operatorname{dist}(x^k, [f \le t]) \to +\infty$. From $\inf f \le t' \le t$, we have $[f \le t'] \subseteq [f \le t]$.

Hence, $\operatorname{dist}\ x^k,\left[f\le t'\right]\ \ge \operatorname{dist}\ x^k,\left[f\le t\right]$. It follows that $\operatorname{dist}\ x^k,\left[f\le t'\right]\ \to+\infty$. This implies that $\{x^k\}$ is a sequence of second type of $\left[f\le t'\right]$, i.e., $t'\in F_+^2$.

Definition 3.5 ([5]). Put $h_+=\begin{cases}\sup\ \ t\in\mathbb{R}:t\in F_+^2\ \ ,F_+^2\neq\varnothing,\\ \inf f \qquad\qquad\qquad\ \ ,\ F_+^2=\varnothing.\end{cases}$ The value h_+ is called *threshold* of global Holderian error bounds of f .

We will extend the following result in [5] from polynomial functions to differentiable semi-algebraic functions.

Theorem 3.6. *We have*

$$\Lambda_+(f)=\begin{cases}(h_+,+\infty)\setminus F_+^1 & \text{if } \ h_+\in F_+^2,\\ [h_+,+\infty)\setminus F_+^1 & \text{if } \ h_+\notin F_+^2,\\ [\inf f,+\infty)\setminus F_+^1 & \text{if } F_+^2=\varnothing,\ \ \inf f>-\infty,\\ \mathbb{R}\setminus F_+^1 & \text{if } F_+^2=\varnothing,\ \ \inf f=-\infty.\end{cases}$$

Proof. By Theorem 3.2, the set $\left[f\le t\right]$ has a GHEB if and only if $t\notin F_+^1\cup F_+^2$. From Proposition 3.4, we see that F_+^2 is the interval. So we have the following cases:

- $F_+^2=\varnothing$ if $h_+=\inf f$;
- $F_+^2=\mathbb{R}$ if $h_+=+\infty$;
- $F_+^2=[\inf f,h_+]$ or $F_+^2=(-\infty,h_+]$ if $h_+\in F_+^2$;
- $F_+^2=[\inf f,h_+)$ or $F_+^2=(-\infty,h_+)$ if $h_+\notin F_+^2$.

Two last items have the assumption $f^{-1}(\inf f)\neq\varnothing$ because if we have converse claim, then we do not take the left end of the set F_+^2. Combining above cases, it follows that the formula of the set $\Lambda_+(f)$.

The relationship between the set $\Lambda_+(f)$ and the set of Fedoryuk values, in the case of f is a differentiable semi-algebraic, is following facts:

1. $F_+^1\subset\tilde{K}_\infty\ \ f$;

Proof. Indeed, put $X=\ \ x\in\mathbb{R}^n\mid f(x)\ge t\ \$, then *X* is a complete metric space with metric is Euclidean metric and the function $f:X\to\mathbb{R}$ is bounded below. Suppose that $t\in F_+^1$ and $\{x^k\}$ is a sequence of the first type of $\left[f\le t\right]$, thus $\left\|x^k\right\|\to\infty, f(x^k)\ge t, f(x^k)\to t,\ \ \exists\delta>0$ such that

$$\operatorname{dist}\ x^k,\left[f\le t\right]\ \ge\delta>0.$$

Put $\varepsilon_k=f(x^k)-t$, we have $\varepsilon_k>0$ and $\varepsilon_k\to0$ (as $k\to+\infty$). Set $\lambda_k=\sqrt{\varepsilon_k}$, by using The Ekeland Variational Principle (for instance, see [6]), it follows that a sequence $\{y^k\}\subset X$ such that

$$f(y^k)\le t+\varepsilon_k=f(x^k) \text{ and } \operatorname{dist}\ y^k,x^k\ \ \le\lambda_k,$$

besides, for any $x \in X, x \neq y^k$, we have

$$f(x) \geq f(y^k) - \frac{\varepsilon_k}{\lambda_k} \operatorname{dist}(x, y^k). \qquad (2)$$

Since $\operatorname{dist}(y^k, x^k) \leq \lambda_k = \sqrt{\varepsilon_k} \to 0$ and $\operatorname{dist}(x^k, [f \leq t]) \geq \delta > 0$, then the ball $B\left(y^k, \frac{\delta}{2}\right) = \left\{x \in \mathbb{R}^n : \operatorname{dist}(y^k, x) \leq \frac{\delta}{2}\right\}$ is in the set *X*. By (2), we have $\frac{f(y^k + \tau u) - f(y^k)}{\tau} \geq -\sqrt{\varepsilon_k}$ for all $u \in \mathbb{R}^n, \|u\| = 1$ and $\tau \in \left[0, \frac{\delta}{2}\right)$. Take $\tau \to 0$, we have $\left\langle \nabla f(y^k), u \right\rangle \geq -\sqrt{\varepsilon_k}$. Put $u = -\frac{\nabla f(y^k)}{\|\nabla f(y^k)\|}$, then we get $\|\nabla f(y^k)\| \leq \sqrt{\varepsilon_k}$. Combine $f(y^k) \to t$ and $\|y^k\| \to \infty$, we get $t \in \tilde{K}_\infty(f)$.

2. If $[f \leq t]$ has a sequence of the second type, then there exists another sequence of the second type (y^k) of $[f \leq t]$ which satisfies the following properties: $\|\nabla f(y^k)\| \to 0$ and $\lim_{k \to \infty} f(y^k) \in \tilde{K}_\infty(f)$. Specifically, $[t, M] \cap \tilde{K}_\infty(f) \neq \varnothing$, where $M := \sup_k (f(y^k))$.

3. If $h_+ \neq \pm\infty$, then $h_+ \in \tilde{K}_\infty(f)$.

The following theorem give an answer for the question when the set is $\Lambda_+(f)$ non-empty.

Theorem 3.7. *If the Fedoryuk set $\tilde{K}_\infty(f)$ is a finite set, then $\Lambda_+(f) \neq \varnothing$.*

Proof. (It is like proof of Theorem 4.1 in [5]).

Suppose that $\Lambda_+(f) = \varnothing$. By assumption the Fedoryuk set is non-empty and property $F_+^1 \subset \tilde{K}_\infty(f)$, F_+^1 is also a finite set. Hence, from the formula of $\Lambda_+(f)$, we have $\Lambda_+(f) = \varnothing$ if and only if $h_+ = +\infty$. Therefore, where $t_1 \in (\inf f, +\infty)$, there exists a sequence of the second type of $[f \leq t_1]$. By the Property 2, there exist $M_1 > t_1$ and $a_1 \in [t_1, M_1] \cap \tilde{K}_\infty(f)$. Take t_2 such that $M_1 < t_2$. Since $h_+ = +\infty$, then there exists a sequence of the second type of $[f \leq t_2]$. Hence there exists $M_2 > t_2$ and a_2 such that $a_2 \in [t_2, M_2] \cap \tilde{K}_\infty(f)$. By recurrence, we can find an infinite sequence $(a_n)_n$ satisfies $a_1 < a_2 < \ldots < a_n < \ldots$ belong to $\tilde{K}_\infty(f)$. Hence, $\#\tilde{K}_\infty(f) = +\infty$. This is the contradiction.

2.3. The case of differentiable semi-algebraic function in two variables

The following corollary is the case of f is a polynomial function in two variables.

Corollary 3.8. (Theorem 6.1 in [5]) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function in two variables, then $\Lambda_+(f) \neq \varnothing$.*

This case has some different facts from differentiable semi-algberaic functions. In the case of polynomial functions, the Fedoryuk set $\tilde{K}_\infty(f)$ is always finite set in the case of two variables. Therefore, we have the above corollary. But in the case of differentiable semi-algebraic functions, even though two variables, the Fedoryuk set can be infinite. Indeed, let us consider the following examples.

Example 3.9. ([7]) Consider the following differentiable semi-algebraic function $f(x,y)=\dfrac{y}{1+x^2}$.

Take a sequence $(x^k)_k$ such that $x^k=(k, a(1+k^2)), a\in\mathbb{R},$ then we have $\|x^k\|\to\infty, f(x^k)=a, \nabla f(x^k)\to 0.$ This implies that $\tilde{K}_\infty(f)=\mathbb{R}$.

Example 3.10. Consider the following differentiable semi-algebraic function $g(x,y)=\dfrac{y^2}{1+x^2}$.

Take a sequence $(x^k)_k$ such that $x^k=(k,\sqrt{a(1+k^2)}), a\in\mathbb{R}, a\geq 0,$ then we have $\|x^k\|\to\infty, g(x^k)=a, \nabla g(x^k)\to 0.$ Hence $\tilde{K}_\infty(g)=[0,+\infty)$.

Two above examples show that the set of Fedoryuk values can be infinite in the case of f is a differentiable semi-algebraic function.

3. Conclusions

The main results here, i.e., Theorem 3.6 and Theorem 3.7, are two extensions of the results in the paper [5] by Huy Vui Hà and Phi Dung Hoàng, which answer the above question, that is when the set is non-empty in the case that f is a semi-differentiable algebraic function. In addition, the examples 3.9 and 3.10 clearly show the difference in the assumption of Theorem 3.7, that is, the set of Fedoryuk values in the case of a two variables semi-algebraic function can be an infinite set, from which it is possible to set $\Lambda_+(f)$ is not non-empty as in the case where f is a polynomial function.

Acknowledgments

The author thanks the anonymous referees for the careful reading and constructive comments on an earlier version of this paper.

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