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On the tangent sets of constraint systems

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Abstract

In this paper, we present some new results on chain rules for the first-order tangent cone and the second-order tangent set of constraint systems under the assumption on the calmness of the constraint set mapping with canonical perturbation. The obtained results improve and extend the corresponding results in [3], [11], and [12].

Keywords: Calmness, metric subregularity, first-order tangent cone, second-order tangent set, chain rule.

1. Introduction

The investigation of optimality conditions is one of the most attractive topics in optimization theory. In order to derive optimality conditions for a local minimum point $\bar{x}$ of a constrained optimization problem, we must examine ways to perturb $\bar{x}$ while remaining in the constraint system Σ of the considered problem. One of the fundamental concept in this analysis is that of a tangent direction. Tangent directions are crucial for developing optimality conditions for nonlinear optimization problems. More specifically, if a point $\bar{x}$ is a local minimum of the following problem

$$\min_{x\in\Sigma} f(x),$$

where $\Sigma\subset\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}$ is a Fréchet differentiable on $\mathbb{R}^n$, then $\bar{x}$ satisfies the following relation

$$-\nabla f(\bar{x})\in\left[T(\Sigma,\bar{x})\right]^*,$$

where $T(\Sigma,\bar{x})$ is the set of all tangent directions to Σ at $\bar{x}$ and the superscript* denoting the polar cone of $T(\Sigma,\bar{x})$, i.e.,

$$\left[T(\Sigma,\bar{x})\right]^* = \left\{u\in\mathbb{R}^n : \langle u,v\rangle\le 0 \quad \forall v\in T(\Sigma,\bar{x})\right\}.$$

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All optimality conditions, in one way or another, decrypt this inclusion. In general, the sets of tangent directions may be nonconvex, which makes the analysis of optimality conditions difficult.

For a general optimization problem, the constraint set usually has the following form

$$\Sigma = \left\{ x \in \Omega : g\left(x\right) \in K \right\},$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a Fréchet differentiable mapping, $\Omega \subset \mathbb{R}^n$, $K \subset \mathbb{R}^m$ are closed subsets. It is easy to prove that (see Theorem 3.1 below)

$$T\left(\Sigma,\bar{x}\right) \subset \left\{ v \in T\left(\Omega,\bar{x}\right) : \nabla g\left(\bar{x}\right) v \in T\left(K, g\left(\bar{x}\right)\right) \right\}.$$

In general, the equality in the above relation is not guaranteed, unless the constraint system satisfies a constraint qualification condition. In [12, Theorem 6.31], Rockafellar and Wets proved that the equality holds under the metric regularity of the constraint system Σ. Recently, Mohammadi et al. [11, Proposition 4.2] have shown that if $\Omega = \mathbb{R}^n$, then the equality also holds under the (very weak) metric subregularity constraint qualification.

In this paper, we present some new results on chain rules for the first-order tangent cone and the second-order tangent set of constraint systems, which improve and extend the corresponding results in [3], [11], and [12].

2. Materials and methods or Experiments

2.1. Preliminaries

2.1.1. Tangents and normals

Let Σ be a nonempty closed subset in $\mathbb{R}^n$ and $\bar{x} \in \Sigma$.

Definition 2.1. (i) The *regular/Fréchet normal cone* $N\left(\bar{x}, \Sigma\right)$ to Σ at $\bar{x}$ is

$$N\left(\bar{x},\Sigma\right) := \left\{ v \in \mathbb{R}^n : \langle v, x-\bar{x}\rangle \le o\left(\| x-\bar{x} \|\right) \quad \text{as } x \to \bar{x} \ \text{ with } \ x \in \Sigma \right\}.$$

(ii) The *limiting/Mordukhovich normal cone* $N\left(\bar{x},\Sigma\right)$ to Σ at $\bar{x}$ consists of all vectors $v \in \mathbb{R}^n$ such that there exist sequences $x^k \xrightarrow{\Sigma} \bar{x}$ and $v^k \to v$ with $v^k \in N\left(x^k, \Sigma\right)$ as $k \to \infty$.

It follows directly from the definition of normal cones that they always satisfy the following inclusions

$$N\left(\Sigma,\bar{x}\right) \subset N\left(\Sigma,\bar{x}\right),$$
$$N\left(\Sigma_1 \cap \Sigma_2,\bar{x}\right) \supset N\left(\Sigma_1,\bar{x}\right) + N\left(\Sigma_2,\bar{x}\right).$$

We say that the set Σ is *normally regular* at $\bar{x} \in \Sigma$ if $N\left(\bar{x},\Sigma\right) = N\left(\bar{x},\Sigma\right)$. As shown in [10, Proposition 1.5], if the set Σ is locally convex around $\bar{x}$, i.e., there exists a neighborhood U of $\bar{x}$ such that $\Sigma \cap U$ is convex, then Σ is normally regular at $\bar{x}$ and the normal cone to Σ at $\bar{x}$ reduces to the normal cone in the sense of convex analysis, i.e,

$$N\left(\bar{x},\Sigma\right) = \left\{ x^* \in \mathbb{R}^n : \ \langle x^*, x-\bar{x}\rangle \le 0, \ \text{for all } \ x \in \Sigma \cap U \right\}.$$

We now present an example that there exists a normally regular set but not locally convex. For this task, we recall the concept of a locally convex function. A function $f:\mathbb{R}^n \to \mathbb{R}$ is called locally convex around $\bar{x}$ if there is a $\delta>0$ such that $f|_{B(\bar{x},\delta)}$ is convex. It is easy to check that the function f is locally convex around $\bar{x}$ if and only if its epigraph is locally convex at $(\bar{x}, f(\bar{x}))$.

Example 2.2. Let the function $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) = x_1^3 + x_2^3 + x_1^2 x_2^2 + x_2^2$ for all $x=(x_1,x_2)\in\mathbb{R}^2$. We claim that f is not locally convex around $\bar{x}=(0,0)$. Indeed, let $x^1=(\delta,0)^T$ and $x^2=(-\delta,0)^T$ for some $\delta>0$. Then for $\lambda\in(0,1)$ we have $f(x^1)=\delta^3$, $f(x^2)=-\delta^3$ and $f(\lambda x^1+(1-\lambda)x^2)=\delta^3(2\lambda-1)^3$. Hence,

$$f(\lambda x^1+(1-\lambda)x^2) > \lambda f(x^1)+(1-\lambda) f(x^2)$$

whenever $(2\lambda-1)^3 > 2\lambda-1$ and which is true for all $0<\lambda<\frac{1}{2}$. This implies that f is not locally convex around $\bar{x}$. Consequently, $\operatorname{epi} f$ is not locally convex at $(\bar{x}, f(\bar{x}))$. We now show that $\operatorname{epi} f$ is normally regular at $(\bar{x}, f(\bar{x}))$. Indeed, we have

$$\operatorname{epi} f = \{(x,\lambda)\in\mathbb{R}^2\times\mathbb{R} : f(x)\le\lambda\} = \{(x,\lambda)\in\mathbb{R}^2\times\mathbb{R} : f(x)-\lambda\le 0\} = \varphi^{-1}(\mathbb{R}_-),$$

where the function $\varphi:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is defined by $\varphi(x,\lambda)=f(x)-\lambda$ for all $(x,\lambda)\in\mathbb{R}^2\times\mathbb{R}$. Clearly, φ is strictly differentiable on $\mathbb{R}^2\times\mathbb{R}$ and since $\nabla\varphi(\bar{x}, f(\bar{x}))=(0,0,-1)^T\neq 0$, $\nabla\varphi(\bar{x}, f(\bar{x}))$ is surjective. Thanks to [10, Theorem 1.19]

(about the normal regularity of inverse images under strictly differentiable mappings) and the convexity of $\mathbb{R}_-$, $\operatorname{epi} f=\varphi^{-1}(\mathbb{R}_-)$ is normally regular around $(\bar{x}, f(\bar{x}))$.

Proposition 2.3. *Let* $f:\mathbb{R}^n\to\mathbb{R}$ *be an arbitrary function and* $\bar{x}\in\mathbb{R}^n$. *If* f *is strictly differentiable at* $\bar{x}$, *then* $\operatorname{epi} f$ *is normally regular around* $(\bar{x}, f(\bar{x}))$.

Proof. Let the function $\varphi:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ be defined by $\varphi(x,\lambda)=f(x)-\lambda$ for all $(x,\lambda)\in\mathbb{R}^n\times\mathbb{R}$. Then, we have

$$\operatorname{epi} f = \{(x,\lambda)\in\mathbb{R}^n\times\mathbb{R} : f(x)\le\lambda\} = \{(x,\lambda)\in\mathbb{R}^n\times\mathbb{R} : f(x)-\lambda\le 0\} = \varphi^{-1}(\mathbb{R}_-).$$

Clearly, φ is strictly differentiable on $\mathbb{R}^n\times\mathbb{R}$ and since $\nabla\varphi(\bar{x}, f(\bar{x}))=\begin{pmatrix}\nabla f(\bar{x})\\ -1\end{pmatrix}\neq 0$, $\nabla\varphi(\bar{x}, f(\bar{x}))$ is surjective. Thanks to [10, Theorem 1.19] and the convexity of $\mathbb{R}_-$, $\operatorname{epi} f=\varphi^{-1}(\mathbb{R}_-)$ is normally regular around $(\bar{x}, f(\bar{x}))$.

Definition 2.4. (i) The *Bouligand tangent/contingent cone* $T(\Sigma,\bar{x})$ to Σ at $\bar{x}$ is

$$T(\Sigma,\bar{x})=\{v\in\mathbb{R}^n : \exists t_k\downarrow 0, \exists v^k\to v,\ \bar{x}+t_k v^k\in\Sigma\ \ \forall k\in\mathbb{N}\}.$$

(ii) The *regular/Clarke tangent cone* $T(\Sigma,\bar{x})$ to Σ at $\bar{x}$ is

$$T(\Sigma,\bar{x}) = \left\{ v \in \mathbb{R}^n \ : \ \forall x^k \xrightarrow{\Sigma} \bar{x},\ \forall t_k \downarrow 0,\ \exists v^k \to v,\ x^k + t_k v^k \in \Sigma \ \ \forall k \in \mathbb{N} \right\}.$$

It is well-known that $T(\Sigma,\bar{x})$ is a closed cone, $T(\Sigma,\bar{x})$ is a closed and convex cone, and $T(\Sigma,\bar{x}) \subset T(\Sigma,\bar{x})$. Moreover, by [12, Theorem 6.28], we have

$$N(\Sigma,\bar{x}) = \left[T(\Sigma,\bar{x})\right]^* \ \text{ and } \ T(\Sigma,\bar{x}) = \left[N(\Sigma,\bar{x})\right]^*.$$

Definition 2.5. The *second-order tangent set* $T^2(\Sigma,\bar{x},v)$ to Σ at $\bar{x}$ for $v \in \mathbb{R}^n$ is

$$T^2(\Sigma,\bar{x},v) = \left\{ w \in \mathbb{R}^n : \forall t_k \downarrow 0, \exists w^k \to w, \bar{x} + t_k v + \frac{1}{2} t_k^2 w^k \in \Sigma \ \ \forall k \in \mathbb{N} \right\}.$$

By definition, it is easy to check that $u \in T^2(\Sigma,\bar{x},v)$ if and only if

$$d\left(\bar{x} + tv + \frac{1}{2} t^2 u, K \right) = o\left(t^2\right).$$

Moreover, the mappings $T(\cdot,\bar{x})$ and $T^2(\cdot,\bar{x},v)$ are isotone, i.e., if $\Sigma_1 \subset \Sigma_2$, then

$$T(\Sigma_1,\bar{x}) \subset T(\Sigma_2,\bar{x}) \ \text{ and } \ T^2(\Sigma_1,\bar{x},v) \subset T^2(\Sigma_2,\bar{x},v).$$

It is well-known that the set $T^2(\Sigma,\bar{x},v)$ is closed but it may be empty. Moreover, if $T^2(\Sigma,\bar{x},v)$ is nonempty, then $v \in T(\Sigma,\bar{x})$. When Σ is a polyhedral set then

$$T^2(\Sigma,\bar{x},v) = T\left(T(\Sigma,\bar{x}),v\right).$$

2.1.2. The calmness

Definition 2.6. Let $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a multifunction and $(\bar{y},\bar{x}) \in \text{gph} M$. We say that:

(i) M is *pseudo-Lipschitz/Lipschitz-like/having Aubin property* at $(\bar{y},\bar{x})$ (or, synonymously, the inverse multifunction M^{-1} *is metrically regular* (MRCQ) at $(\bar{x},\bar{y})$) if there exist $\kappa,\delta > 0$ such that

$$d\left(x, M\left(y^1\right)\right) \le \kappa d\left(y^1,y^2\right) \ \ \forall x \in M\left(y^2\right) \cap B(\bar{x},\delta), \forall y^1, y^2 \in B(\bar{y},\delta).$$

(ii) M is *calm* at $(\bar{y},\bar{x})$ (or, synonymously, the inverse multifunction M^{-1} *is metrically subregular* (MSCQ) at $(\bar{x},\bar{y})$) if there exist $\kappa,\delta > 0$ such that

$$d\left(x, M(\bar{y})\right) \le \kappa d(y,\bar{y}) \ \ \forall x \in M(y) \cap B(\bar{x},\delta), \forall y \in B(\bar{y},\delta).$$

From definition, the calmness of a multifunction is strictly weaker than the pseudo-Lipschitz property. The late property admits a complete pointwise characterization via the coderivative criterion. As shown in [9], a closed multifunction $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is pseudo-Lipschitz at $(\bar{y},\bar{x}) \in \text{gph} M$ if and only if

$$D^* M(\bar{y},\bar{x})(0) = \{0\},$$

where $D^* M(\bar{y},\bar{x})$ refers to Mordukhovich's coderivative of M at $(\bar{y},\bar{x})$, i.e.,

$$D^* M(\bar{y},\bar{x})(v) := \left\{ u \in \mathbb{R}^m : (u,-v) \in N\left(\text{gph} M, (\bar{y},\bar{x})\right) \right\}, \ v \in \mathbb{R}^n.$$

Calmness/metric subregularity was used as a qualification condition in the theory of subdifferential calculus and necessary optimality conditions; see, e.g., [7, 8, 10, 12]. This property also plays an important role in the theory of weak sharp minima and error bounds; see, e.g., [1, 2, 6]. Despite the intensive applications of the calmness, to the best of our knowledge, so far there have been no pointwise characterization for this property.

2.2. Main results

In this section, we consider the following constraint system

$$\Sigma = \{x \in \Omega : g(x) \in K\}, \tag{1}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous mapping, $\Omega \subset \mathbb{R}^n$, $K \subset \mathbb{R}^m$ are closed subsets. The constraint set mapping with canonical perturbation $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ associated with the system (1) is defined as follows

$$M(y) = \{x \in \Omega \,:\, g(x) + y \in K\}. \tag{2}$$

Clearly, $M(y) = \Omega \cap g^{-1}(K - y)$ for all $y \in \mathbb{R}^m$ and $M(0) = \Omega \cap g^{-1}(K) = \Sigma$. As shown in [6], the calmness of M at $(0, \bar{x}) \in \operatorname{gph} M$ is equivalent to the following metric qualification condition: there exist $\gamma > 0$ and such that $\delta > 0$

$$d(x, \Sigma) \le \gamma\big(d(x, \Omega) + d(g(x), K)\big) \ \ \forall x \in B(\bar{x}, \delta),$$

provide that g is locally Lipschitz around $\bar{x}$. Some sufficient conditions for the calmness of the multifunction M can be found in [4].

The following result gives first-order chain rules for tangents and normals to the constraint system (1) under the calmness of M.

Theorem 2.7. *Assume that g is Fréchet differentiable at $\bar{x} \in \Sigma$, Ω and K are normally regular at $\bar{x}$ and $g(\bar{x})$, respectively, and M is calm at $(0, \bar{x})$. Then the following equalities hold*

$$N(\Sigma, \bar{x}) = \nabla g(\bar{x})^T N(K, g(\bar{x})) + N(\Omega, \bar{x}), \tag{3}$$

$$T(\Sigma, \bar{x}) = \{v \in T(\Omega, \bar{x}) \,:\, \nabla g(\bar{x}) v \in T(K, g(\bar{x}))\}. \tag{4}$$

Proof. We first prove the equality (3). It follows from [6, Proposition 3.4] and the fact that $D^* g(\bar{x})(y^*) = \nabla g(\bar{x})^T (y^*)$ for all $y^* \in \mathbb{R}^n$ that

$$\begin{aligned} N(\Sigma, \bar{x}) &\subset \bigcup_{y^* \in N(K, g(\bar{x}))} D^* g(\bar{x})(y^*) + N(\Omega, \bar{x}) \\ &= \nabla g(\bar{x})^T N(K, g(\bar{x})) + N(\Omega, \bar{x}). \end{aligned}$$

In the other hand, by the normal regularity of two sets Ω, K and [10, Corollary 1.15], we have

$$\begin{aligned} N(\Sigma, \bar{x}) &= N\big(\Omega \cap g^{-1}(K), \bar{x}\big) \\ &\supset N\big(\Omega \cap g^{-1}(K), \bar{x}\big) \\ &\supset N\big(g^{-1}(K), \bar{x}\big) + N(\Omega, \bar{x}) \\ &\supset \nabla g(\bar{x})^T N(K, g(\bar{x})) + N(\Omega, \bar{x}) \\ &= \nabla g(\bar{x})^T N(K, g(\bar{x})) + N(\Omega, \bar{x}). \end{aligned}$$

and so complete the proof of (3).

We are now in position to prove (4). Let $v \in T(\Sigma,\bar{x})$ then by definition $v \in T(\Omega,\bar{x})$ and there exist sequences $t_k \downarrow 0$ and $v^k \to v$ such that $\bar{x}+t_k v^k \in \Sigma$ for all $k \in \mathbb{N}$. The late inclusion implies that $g(\bar{x}+t_k v^k) \in K$ for all $k \in \mathbb{N}$. By the differentiability of g at $\bar{x}$, we have

$$g(\bar{x}+t_k v^k) = g(\bar{x}) + t_k \nabla g(\bar{x})(v^k) + o(t_k) = g(\bar{x}) + t_k\left[\nabla g(\bar{x})(v^k) + \frac{o(t_k)}{t_k}\right] \in K \ \ \forall k \in \mathbb{N}.$$

This and the fact that $\nabla g(\bar{x})(v^k) + \frac{o(t_k)}{t_k} \to \nabla g(\bar{x})(v)$ imply that $\nabla g(\bar{x})(v) \in T(K, g(\bar{x}))$. Thus

$$T(\Sigma,\bar{x}) \subset \left\{v \in T(\Omega,\bar{x}) : \nabla g(\bar{x}) v \in T(K, g(\bar{x}))\right\}.$$

To prove the inverse inclusion, we get from (3) and the normal regularity of Ω at $\bar{x}$ that

$$\begin{aligned} T(\Sigma,\bar{x}) &= \left[N(\Sigma,\bar{x})\right]^* = \left[\nabla g(\bar{x})^T N(K, g(\bar{x})) + N(\Omega,\bar{x})\right]^* \\ &= \left[\nabla g(\bar{x})^T N(K, g(\bar{x}))\right]^* \cap \left[N(\Omega,\bar{x})\right]^* \\ &= \left[\nabla g(\bar{x})^T N(K, g(\bar{x}))\right]^* \cap T(\Omega,\bar{x}) \\ &= \left[\nabla g(\bar{x})^T N(K, g(\bar{x}))\right]^* \cap T(\Omega,\bar{x}). \end{aligned}$$

We claim that

$$\left\{v \in \mathbb{R}^n : \nabla g(\bar{x}) v \in T(K, g(\bar{x}))\right\} \subset \left[\nabla g(\bar{x})^T N(K, g(\bar{x}))\right]^*.$$

Indeed, let $v \in \mathbb{R}^n$ such that $\nabla g(\bar{x}) v \in T(K, g(\bar{x}))$. It follows from the normal regularity of K at $g(\bar{x})$ that

$$T(K, g(\bar{x})) = T(K, g(\bar{x})) = \left[N(K, g(\bar{x}))\right]^*.$$

Hence, for any $d \in N(K, g(\bar{x}))$ we have

$$\left\langle v, \nabla g(\bar{x})^T d\right\rangle = \left\langle \nabla g(\bar{x}) v, d\right\rangle \le 0.$$

This implies that $v \in \left[\nabla g(\bar{x})^T N(K, g(\bar{x}))\right]^*$, as required. Therefore

$$\begin{aligned} T(\Sigma,\bar{x}) \supset T(\Sigma,\bar{x}) &= \left[\nabla g(\bar{x})^T N(K, g(\bar{x}))\right]^* \cap T(\Omega,\bar{x}) \\ &\supset \left\{v \in \mathbb{R}^n : \nabla g(\bar{x}) v \in T(K, g(\bar{x}))\right\} \cap T(\Omega,\bar{x}) \\ &= \left\{v \in T(\Omega,\bar{x}) : \nabla g(\bar{x}) v \in T(K, g(\bar{x}))\right\}. \end{aligned}$$

The proof is complete. □

Remark 2.8. In [12, Theorems 6.14 and 6.31], Rockafellar and Wets proved that the equalities (3) and (4) hold under the following constraint qualification

$$\left.\begin{array}{l} y^* \in N\left(K, g\left(\bar{x}\right)\right) \\ -\nabla g\left(\bar{x}\right)^T\left(y^*\right) \in N\left(\Omega, \bar{x}\right) \end{array}\right\} \Rightarrow y^* = 0. \tag{5}$$

Actually, the constraint qualification (5) is equivalent to the pseudo-Lispchitz of M at $\left(0, \bar{x}\right)$ (or, synonymously, M^{-1} is MRCQ at $\left(\bar{x}, 0\right)$). As shown in [3, Corollary 2.2], this condition is nothing more than the Robinson constraint qualification (RCQ), i.e.,

$$0 \in \operatorname{int}\left[\nabla g\left(\bar{x}\right)\left(\Omega - \bar{x}\right) - \left(K - g\left(\bar{x}\right)\right)\right]. \tag{6}$$

Furthermore, when $K = \mathbb{R}^p_+ \times \left\{0_{\mathbb{R}^q}\right\}$, $\Omega = \mathbb{R}^n$ then the RCQ reduces to the classical Mangasarian-Fromowitz constraint qualification; see, e.g., [3, 10, 12]. Since, the pseudo-Lipschitz property is strictly stronger than the calmness, Theorem 2.7 improves the corresponding results in [12, Theorems 6.14 and 6.31]. Moreover, this theorem extends the recent result by Mohammadi et al. [11, Proposition 4.2] where $\Omega = \mathbb{R}^n$.

The following simple example is designed to clarify this remark.

Example 2.9. Let $\Omega = \mathbb{R}_+, K = \mathbb{R}_-$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = x$. Then we see that

$\Sigma = \left\{x \in \Omega \, : \, g\left(x\right) \in K\right\} = \left\{0\right\}$.

Let $\bar{x} = 0 \in \Sigma$. Clearly, Ω and K are convex and so they are normally regular. Moreover, it is easy to check that the multifunction $M\left(y\right) = \Omega \cap g^{-1}\left(K - y\right)$ is calm at $\left(0, \bar{x}\right)$. Thus, by Theorem 2.7, we have

$$\begin{aligned} N\left(\Sigma, \bar{x}\right) &= \nabla g\left(\bar{x}\right)^T N\left(K, g\left(\bar{x}\right)\right) + N\left(\Omega, \bar{x}\right) \\ &= N\left(K, 0\right) + N\left(\Omega, 0\right) = \mathbb{R}_+ + \mathbb{R}_- = \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} T\left(\Sigma, \bar{x}\right) &= \left\{v \in T\left(\Omega, \bar{x}\right) \, : \, \nabla g\left(\bar{x}\right) v \in T\left(K, g\left(\bar{x}\right)\right)\right\} \\ &= T\left(\Omega, \bar{x}\right) \cap T\left(K, g\left(\bar{x}\right)\right) = \mathbb{R}_+ \cap \mathbb{R}_- = \left\{0\right\}. \end{aligned}$$

Clearly, the condition (5) does not satisfy and $\Omega \subsetneqq \mathbb{R}^n$. Thus [11, Proposition 4.2] and [12, Theorems 6.14 and 6.31] cannot be applied for this example.

Remark 2.10. The normal regularity of Ω and K in Theorem 2.7 is essential. This means that the calmness of M alone is not sufficient for the equalities (3) and (4) in case of arbitrary closed sets Ω and K. To see this, let us consider the following example.

Example 2.11. Let $\Omega = \left\{x = \left(x_1, x_2\right) \in \mathbb{R}^2 \, : \, x_2 \geq -x_1\right\}$, $K = \left\{x = \left(x_1, x_2\right) \in \mathbb{R}^2 : \, x_2 \geq -\,|\, x_1 \,|\right\}$, and $g : \mathbb{R}^2 \to \mathbb{R}^2, g\left(x\right) = x$. Then,

$\Sigma = \left\{x = \left(x_1, x_2\right) \in \Omega \, : x \in K\right\} = \Omega \cap K = \Omega$.

Let $\bar{x} = \left(0, 0\right) \in \Sigma$. An easy computation shows that $N\left(K, \bar{x}\right) = \left\{0\right\}$ and

$N\left(K, \bar{x}\right) = \left\{\left(v, v\right) \in \mathbb{R}^2 : v \leq 0\right\} \cup \left\{\left(v, -v\right) \in \mathbb{R}^2 : v \geq 0\right\}$.

Hence, K is not normally regular at $\bar{x}$. Clearly, M is calm at $\left(0, \bar{x}\right)$. However,

$$N(\Sigma,\bar{x}) = N(\Omega,\bar{x}) = \{(v,v) : v \le 0\}$$

and

$$\nabla g(\bar{x})^T N(K, g(\bar{x})) + N(\Omega,\bar{x}) = N(K, g(\bar{x})) + N(\Omega,\bar{x}) = \{(v_1,v_2) \in \mathbb{R}^2 : v_2 \le -|v_1|\}.$$

Thus (3) does not hold.

The following theorem provides a chain rule for second-order tangent sets to the constraint system (1).

Theorem 2.12. *Assume that* g *is twice Fréchet differentiable at* $\bar{x} \in \Sigma$, Ω *and* K *are normally regular at* $\bar{x}$ *and* $g(\bar{x})$, *respectively, and* M *is calm at* $(0,\bar{x})$. *Then, for any* $v \in T(\Sigma,\bar{x})$ *we have*

$$T^2(\Sigma,\bar{x},v) = \{u \in T^2(\Omega,\bar{x},v) : \nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v) \in T^2(K, g(\bar{x}), \nabla g(\bar{x})v)\}. \qquad (7)$$

Proof. The proof follows some ideals of Cominetti [3, Theorem 3.1]. Since $\Sigma \subset \Omega$, we have $T^2(\Sigma,\bar{x},v) \subset T^2(\Omega,\bar{x},v)$. Let $u \in T^2(\Sigma,\bar{x},v)$. By definition, for any $t_k \downarrow 0$ there exists a sequence $u^k \to u$ such that $x^k := \bar{x} + t_k v + \frac{1}{2}t_k^2 u^k \in \Sigma$ for all $k \in \mathbb{N}$. This and the twice differentiability of g at $\bar{x}$ imply that

$$\begin{aligned} K \ni g(x^k) &= g(\bar{x}) + \nabla g(\bar{x})(x^k - \bar{x}) + \frac{1}{2}\nabla^2 g(\bar{x})(x^k - \bar{x}, x^k - \bar{x}) + o(t_k^2) \\ &= g(\bar{x}) + t_k \nabla g(\bar{x})v + \frac{1}{2}t_k^2\left[\nabla g(\bar{x})u^k + \nabla^2 g(\bar{x})(v,v)\right] + o(t_k^2) \end{aligned}$$

for all $k \in \mathbb{N}$. This and the fact that $\nabla g(\bar{x})u^k + \nabla^2 g(\bar{x})(v,v) \to \nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v)$ as $k \to \infty$ imply that $\nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v) \in T^2(K, g(\bar{x}), \nabla g(\bar{x})v)$. Hence,

$$T^2(\Sigma,\bar{x},v) \subset \{u \in T^2(\Omega,\bar{x},v) : \nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v) \in T^2(K, g(\bar{x}), \nabla g(\bar{x})v)\}.$$

Now let u be an arbitrary element belonging to the right-hand side of (7). Since $u \in T^2(\Omega,\bar{x},v)$, for any $t_k \downarrow 0$ there exists a sequence $u^k \to u$ such that $x^k := \bar{x} + t_k v + \frac{1}{2}t_k^2 u^k \in \Omega$ for all $k \in \mathbb{N}$. By the twice differentiabiliy of g at $\bar{x}$, we have

$$\begin{aligned} g(x^k) &= g(\bar{x}) + t_k \nabla g(\bar{x})v + \frac{1}{2}t_k^2\left[\nabla g(\bar{x})u^k + \nabla^2 g(\bar{x})(v,v)\right] + o(t_k^2) \\ &= g(\bar{x}) + t_k \nabla g(\bar{x})v + \frac{1}{2}t_k^2\left[\nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v)\right] + o(t_k^2). \end{aligned}$$

For each $k \in \mathbb{N}$, put $w^k := g(\bar{x}) + t_k \nabla g(\bar{x})v + \frac{1}{2}t_k^2\left[\nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v)\right]$. It follows from $\nabla g(\bar{x})u + \nabla^2 g(\bar{x})(v,v) \in T^2(K, g(\bar{x}), \nabla g(\bar{x})v)$ that $d(w^k, K) = o(t_k^2)$. From the above arguments and

$$d(g(x^k), K) \le d(g(x^k), w^k) + d(w^k, K),$$

we have $d(g(x^k), K) = o(t_k^2)$. Since M is calm at $(0,\bar{x})$, there exist $\gamma > 0$ such that

$$d\left(x^k,\Sigma\right)\le\gamma\left(d\left(x^k,\Omega\right)+d\left(g\left(x^k\right),K\right)\right))\le\gamma d\left(g\left(x^k\right),K\right)$$

for all k large enough. Hence $d\left(x^k,\Sigma\right)=o\left(t_k^2\right)$. This means that $u\in T^2\left(\Sigma,\bar{x},v\right)$, as required. The proof is now complete.

□

Remark 2.13. In [3, Theorem 3.1], Cominetti proved that the equality (7) holds under the Robinson constraint qualification (RCQ). Since the RCQ (6) is equivalent to the pseudo-Lispchitz of M at $\left(0,\bar{x}\right)$ and it is strictly stronger than the calmness of M at $\left(0,\bar{x}\right)$, Theorem 2.12 improves [3, Theorem 3.1]. Furthermore, since $\Omega\subsetneqq\mathbb{R}^n$, our result extends the corresponding results in [11, Theorem 4.5] and [12, Proposition 13.13].

3. Conclusions

In this paper, we have presented some new calculus rules for first-order tangent cones and second-order tangent sets of general constraint systems. In future work, we aim to apply these results to derive necessary optimality conditions for optimization problems with constraint sets of the form (1).

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