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Euclidean distance degree of zero-set of two polynomials

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Abstract

In this note, we recall the study of the Euclidean distance degree of an algebraic set X which is the zero-point set of a polynomial (see [BSW]). Specifically, consider a hypersurface $f = 0$ defined by a general polynomial f with its support and contains the origin i.e 0 ∈ support of f. In the paper [BSW], the authors study about the Euclidean distance degree (EDD) and found that the EDD of this hypersurface is approximately by the mixed volume (MV) of some Newton polytopes.

The main purpose of this note is to study the case that the manifold is defined by two polynomials $f_1(x) = f_2(x) = 0$. We show that the Euclidean distance degree is equal to the solution of the Lagrange multiplier equation. Furthermore, we also find out that the EDD of this variety is not greater than the mixed volume of Newton polytopes of the associated Lagrange multiplier equations.

Keywords: Euclidean distance degree, Newton polytopes, Mixed volume, Critical point, Tangent space

1. Introduction

Given a point $c = (c_1, c_2, ..., c_n)$ in Euclidean space $\mathbb{R}^n$, consider the function $f_c : \mathbb{R}^n \to \mathbb{R}$ defined by $f_c(x) = \sum (x_i - c_i)^2$, $x = (x_1, x_2, ..., x_n)$. Let X be an algebraic set in $\mathbb{R}^n$. Then, with the general point c, the distance function $f_c \mid \mathrm{X}: \mathrm{X} \to \mathbb{R}$, of the function f_c on X has a finite critical point. The number of critical points does not depend on the general point c and is called the Euclidean distance degree of the set X, denoted by EDD (X).

The study of EDD stems from the fact that many models in data science or mechanical

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engineering can be represented as a real algebraic set, leading to the need to solve the problem of finding the *nearest point problem*. ([DHOST, Section 3], [TJD, §3]):

Nearest point problem: In $\mathbb{R}^n$ given the algebraic set X and a point c, find the point c^* of X such that the function f_c (the distance function from c to X) has a minimum at c^*.

One approach to the above problem is to find and examine all critical points of f_c. Then EDD gives us an algebraic quantity that evaluates the complexity of the above optimization problem.

Algebraic sets and polynomial mapping are basic research objects of algebraic geometry, in particular and of mathematics, in general. For algebraic sets, the topic of Euclidean Distance Degree is widely studied and has many applications in areas such as computer vision, geometric modeling and statistics ([AST, DHOST, MRW2, HS, SSN, TJD, W1, W2]).

In the field of *computer vision*, the problem of triangulation has an important role. Specifically, it is a problem of determining a point in space when its image is known through two cameras with the positions of the two cameras and a given shooting angle. In Mathematics, this is the problem of finding the third vertex of a triangle given two vertices and the angle at those two vertices. When the information is obtained with absolute precision this is a trivial problem, but in practice the pixels obtained by the cameras have noise (see [DHOST, MRW1, MRW2, HS, SSN]). Therefore, the problem is to find the point in space that is maximally compatible with the information obtained from the cameras. This is the optimization problem of the distance function mentioned above (*nearest point problem*) and the Euclidean distance degree is the computational complexity of this problem.

We consider $f \in \mathbb{C}[x_1,...,x_m]$ be a polynomial with support $A \subset \mathbb{Z}^m$, so that

$$f(x) = \sum_{a\in A} c_a x^a \qquad (c_a \in \mathbb{C}).$$

The Newton polytope is defined to be the convex ball of the set $\{a \in \mathbb{N}^n : c_a \neq 0\}$ in $\mathbb{R}^n$.

Theorem 1 *If f is a polynomial whose support A contains* 0, *then*

$$EDD(f) \leq MV(P, P_1, P_2, ..., P_n),$$

where P is the Newton polytope of f and P_i is the Newton polytope of $\partial_i f - \lambda(u_i - x_i)$ for $1 \leq i \leq n$. *There is a dense open subset U of polynomials with support A such that when f ∈ U this inequality is an equality and for u ∈ $\mathbb{C}^n$ general, all solutions to $L_{f,u}$ occur without multiplicity.*

The purpose of this note in to study the Euclidean distance degree of the zero-set *M* of two polynomials f_1, f_2. We will estimate the EDD of *M* in terms of the mixed volume of the polytope defined from f_1, f_2.

2. Euclidean Distance Degree of the Zero-set of two polynomials

Throughout this paper, let $f_1, f_2 \in \mathbb{R}[x_1, x_2, x_3]$ be two polynomials such that $\mathrm{M} = \{x \in \mathbb{C}^3 : f_1(x) = f_2(x) = 0\}$ is smooth.

We consider : $\begin{array}{l} \varphi : \mathrm{M} \to \mathbb{R} \\ x \mapsto \|x-u\|^2 \end{array}$ where $\|x-u\|^2$ is the Euclidean norm.

Since $\{$*nearest points*$\} \subset \{$*critical points of* $\varphi\}$ so the number of critical points ≈ EDD ≈ computational complexity of the *nearest point problem*, which implies that EDD(M) is equal to the number of solutions to the following system of equations:

$$\begin{cases} f_1(x) = f_2(x) = 0 \\ d_x\varphi = 0 \end{cases} \text{ where } \begin{cases} d_x\varphi : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{\varphi(x)} \text{ is a tangent map} \\ x \text{ is the critical point} \end{cases}$$

+) Since the set of all tangent vectors $\vec{v}$ at x such that $d_{f(x)}(v) = 0$ is the tangent space of the manifold M at x so first we need to determine the tangent space at point x:

$\mathrm{T}_x\mathrm{M} = \{v_i = \gamma'_i(t) \mid \mathrm{t} = 0\}$ where $\gamma(t) \subset \mathrm{M}$ and $\gamma(0) = x$.

We have $f_1(x) = f_1(\gamma(t)) = 0$ and $f_2(x) = f_2(\gamma(t)) = 0$. Implying that $d_{fi}(x) = d_{fi}(\gamma(t))$.

- $d_{f_1}(x) = d_{f_1}(\gamma(t)) = \dfrac{\partial f_1}{x_1}(\gamma(t)).\gamma'_1(t) + \dfrac{\partial f_1}{x_2}(\gamma(t)).\gamma'_2(t) + \dfrac{\partial f_1}{x_3}(\gamma(t)).\gamma'_3(t)$.

If $\begin{cases} d_{f_1}(x) = 0 \\ t = 0 \end{cases}$ then $\dfrac{\partial f_1}{x_1}(x).v_1 + \dfrac{\partial f_1}{x_2}(x).v_2 + \dfrac{\partial f_1}{x_3}(x).v_3$.

Thus, $\langle \nabla f_1(x), v \rangle = 0$.

- Similarly, $d_{f_2}(x) = d_{f_2}(\gamma(t)) = 0 \Rightarrow \langle \nabla f_2(x), v \rangle = 0$.

Hence, $\mathrm{T}_x\mathrm{M} = \{v \in \mathbb{C}^3 \mid \langle \nabla f_1(x), v \rangle = \langle \nabla f_2(x), v \rangle = 0\}$.

+) Secondly, we need to find the critical points x.

Suppose that $x = (x_1, x_2, x_3)$, x is a critical point if $\langle \nabla f_i(x), v \rangle = 0$ (*i=1,2*) $\Leftrightarrow \langle \nabla \varphi(x), v \rangle = 0$.

This implies that, $\begin{cases} \nabla f_1(x) = 0 & (1) \\ \nabla f_2(x) = 0 & (2) \\ \nabla \varphi(x) = 0 & (3) \end{cases}$.

Equations (1), (2) and (3) have the same set of solutions if and only if the dimension of equation (1),(2) = the dimension of equation (3). Therefore, (3) is a linear combination of (1) and (2).

Thus, there exists $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\nabla \varphi(x) = \lambda_1.\nabla f_1(x) + \lambda_2.\nabla f_2(x)$

Hence, the critical points x are those that satisfy $u - x = \lambda_1.\nabla f_1(x) + \lambda_2.\nabla f_2(x)$.

The *Lagrange multiplier equations* are the following system of 5 polynomial equations in 5 variables $(\lambda_1, \lambda_2, x_1, x_2, x_3)$.

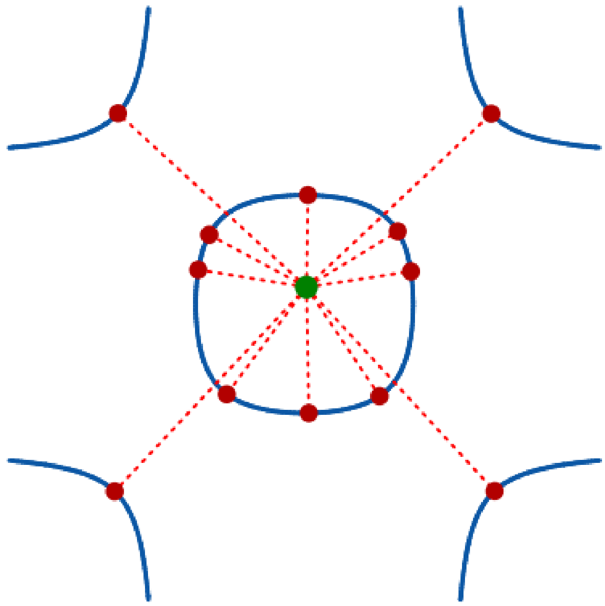
$$L_{f,u}(\lambda, x) := \left\{ (\lambda, x) \in \mathbb{C}_x^3 \times \mathbb{C}_\lambda^2 : \begin{cases} f_1(x) = f_2(x) = 0 \\ u - x = \lambda_1.\nabla f_1(x) + \lambda_2.\nabla f_2(x) \end{cases} \right\}$$

where λ_1, λ_2 are the auxiliary variables (the Lagrange multiplier).

We consider the number of complex solutions of $L_{f,u}(\lambda, x) = 0$. For general *u*, this number is called the Euclidean distance degree (EDD) of the zero-set $\mathrm{M} = \{x \in \mathbb{C}^3 : f_1(x) = f_2(x) = 0\}$: EDD(M) := number of solutions to $L_{f,u}(\lambda, x) = 0$ in $\mathbb{C}^5$ for general *u*.

Here, "general" means for all *u* outside some algebraic set, i.e. outside a set with measure zero.

Example: Let $X = \{(x_1, x_2) : x_1^2 x_2^2 - 3x_1^2 - 3x_2^2 + 5 = 0\} \subset \mathbb{R}^2$ is in blue and u = $(0.025, 0.2)$ is in green. The 12 red points are the critical points of the distance function d_X ; that is, they are the *x*-values of the solutions to $L_{f,u}(\lambda, x) = 0$. In this example, the Euclidean distance degree of *X* is 12, so all complex solutions are in fact real. (Color figure online)



3. Bernstein's Theorem

Bernstein's theorem expresses the relationship between the number of solutions to a polynomial system and the mixed volume.

Theorem 2 (see [BSW]) *Let* $f_1, \ldots, f_m \in \mathbb{C}[x_1, x_2, \ldots, x_m]$ *denote* m *polynomials with*

Newton polytopes $Q_1, Q_2, ..., Q_m$. *Let* $(\mathbb{C}^{\times})^m$ *denote the complex torus of* m*-tuples of nonzero complex numbers and* $\#\mathrm{N}(f_1,..., f_m)$ *denote the number of isolated solutions to* $f_1 = f_2 = ... = f_m = 0$ *in* $(\mathbb{C}^{\times})^m$*, counted by their algebraic multiplicities. Bernstein's theorem states that*

$$\#\mathrm{N}(f_1,..., f_m) \le MV(Q_1,...,Q_m), \quad (1)$$

and the inequality becomes an equality when each f_i *is general.*

The domain is limited to $(\mathbb{C}^{\times})^m$ because Bernstein's theorem concerns Laurent polynomials, in which the exponents of a monomial can be negative.

Assume that all polynomials $f_1,..., f_m$ have the same Newton polytope. This implies that $Q_1,...,Q_m$. For this single polytope, we write *Q*. The mixed volume in (1) then becomes

$$MV(Q_1,...,Q_m) = \mathrm{m}!\mathrm{Vol}(Q),$$

where Vol(*Q*) is the *m*-dimensional Euclidean volume of *Q*.

According to Kushnirenko's theorem (see [KO]) , if $f_1,..., f_m$ are *general* polynomials with Newton polytope *Q*, then $\#\mathrm{N}(f_1,..., f_m) = \mathrm{m}!\mathrm{Vol}(Q)$ (Q).

4. Main Result

This is the main theorem of this paper.

Theorem 3 *Let* $f_1, f_2 \in \mathbb{R}[x_1, x_2, x_3]$ *be two polynomials. If the support* $\mathscr{H}$ *of the polynomials* f_1, f_2 *contains 0, then*

$$EDD(f_1, f_2) \le MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'}),$$

where P_1, P_2 *are the Newton polytopes of* f_1, f_2 *and* $P_{i'}$ *is the Newton polytopes of* $u - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2$ *for* $i = 1,2,3$.

To prove Theorem 3, we need the following result:

Theorem 4 *Let* $\sum := \left\{(a_{\alpha_1},...,a_{\alpha_N}, x_1,...,x_n) \in \mathbb{C}^N \times \left(\mathbb{C}^{\times}\right)^n\right\}$. Let Ω *be the set of critical values of* $p : \Sigma \to \mathbb{C}^N$. *If* $f(a_{\alpha_1},...,a_{\alpha_N}) \in \Omega$ *then* $p^{(-1)}(0)$ *has no singularity in* $\left(\mathbb{C}^{\times}\right)^n$.

Suppose that $p(x_1, x_2, ..., x_n) = \sum_{i=1,..,N} a_{\alpha_i} x^{\alpha_i}$ is a polynomial. If $\{a_{\alpha_i}\}$ is general, then the solution of the polynomial *p(x)* lies in $\left(\mathbb{C}^{\times}\right)^n$:

$$\{x \in \left(\mathbb{C}^{\times}\right)^n : \partial_1 p(x) = ... = \partial_n p(x) = p(x) = 0\}.$$

Considering the equation $\sum_{i=1,..,N} a_{\alpha_i} x^{\alpha_i} = 0$, the equation has a solution because:

+) It is obvious that $\sum_{i=1,..,N} a_{\alpha_i} x^{\alpha_i}$ is exponential, $x_i \in \left(\mathbb{C}^{\times}\right)^n$ and $\sum_{i=1,..,N} a_{\alpha_i} x^{\alpha_i}$ has partial derivatives therefore, $\sum_{i=1,..,N} a_{\alpha_i} x^{\alpha_i}$ is smooth.

+) Let π be a projection :

$$\begin{aligned} \pi: \quad & \Sigma \longrightarrow \mathbb{C}^N \\ & (a_{\alpha_1},...,a_{\alpha_N}, x_1,...,x_n) \mapsto (a_{\alpha_1},...,a_{\alpha_N}) \end{aligned}$$

then $$\begin{aligned} d\pi: \quad & T_x\Sigma \longrightarrow \mathbb{C} \\ & v = \varphi'(t) \mid t=0 \mapsto \pi(\varphi'(t))=0 \end{aligned}.$$

Write $\sum := \left\{(a_{\alpha_1},...,a_{\alpha_N}, x_1,...,x_n) \in \mathbb{C}^N \times \left(\mathbb{C}^{\times}\right)^n\right\}$.

We have : $\Sigma(x) = \Sigma(\varphi(t)) \Rightarrow d\Sigma(x) = d\Sigma(\varphi(t)) = 0$

$$\Leftrightarrow \frac{\partial \Sigma}{a_{\alpha_1}}\varphi(t).\varphi_1'(t) + ... + \frac{\partial \Sigma}{a_{\alpha_N}}\varphi(t).\varphi_N'(t) + \frac{\partial \Sigma}{x_1}\varphi(t).\varphi_{1'}'(t) + ... + \frac{\partial \Sigma}{x_n}\varphi(t).\varphi_{n'}'(t) = 0 \quad (\forall t)$$

$$\Leftrightarrow \frac{\partial \Sigma}{a_{\alpha_1}}(x).v_1 + ... + \frac{\partial \Sigma}{a_{\alpha_N}}(x).v_N + \frac{\partial \Sigma}{x_1}(x).v_{1'} + ... + \frac{\partial \Sigma}{x_n}(x).v_{n'} = 0$$

$$\Leftrightarrow \left\langle \nabla \Sigma(x), v\right\rangle = 0.$$

If $d\pi(v) = 0$ then $\left\langle \nabla \pi(x), v\right\rangle = 0$. For x to be the critical point, then $\left\langle \nabla \Sigma(x), v\right\rangle = 0$ and $\left\langle \nabla \pi(x), v\right\rangle = 0$.

$$\text{Therefore, } \sum = \{(a,x) \in \mathbb{C}^N \times \left(\mathbb{C}^{\times}\right)^n : \text{rank}\begin{pmatrix} x^{\alpha_1} & ... & x^{\alpha_N} & \frac{\partial p}{\partial x_1} & ... & \frac{\partial p}{\partial x_n} \\ 1 & ... & 0 & 0 & ... & 0 \\ 0 & 1... & 0 & 0 & ... & 0 \\ 0 & ... & 1 & 0 & ... & 0 \end{pmatrix} = 1\}$$

$$= \{(a,x) \in \mathbb{C}^N \times \left(\mathbb{C}^{\times}\right)^n : \frac{\partial p}{\partial x_i} = 0\}.$$

Thus, $(a_{\alpha_1},...,a_{\alpha_N})$ is a critical value if and only if $p^{(-1)}(0)$ has no singularity in $\left(\mathbb{C}^{\times}\right)^n$. □

Proof (proof of Theorem 3)

According to *Theorem 4*, the polynomials $f_1, f_2 \in \mathbb{C}[x_1, x_2, x_3]$ are *general* given its support $\mathscr{H}$ and contains 0. We suppose that $u \in \mathbb{C}^n \setminus \mathrm{N}(f_1, f_2)$ is *genaral*, where $\mathrm{N}(f_1, f_2) := \{x \in \mathbb{C}^3 \mid f_i = f_2 = 0\}$.

By Bernstein's theorem, the Lagrange multiplier equations $L_{f,u}(\lambda, x) = 0$ has

$MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'})$ solutions in $\left(\mathbb{C}^{\times}\right)^5$.We need to prove that the Lagrange multiplier equation has no solutions outside $\left(\mathbb{C}^{\times}\right)^5$, which means all the solutions of $L_{f,u}(\lambda, x) = 0$ must lie in $\left(\mathbb{C}^{\times}\right)^5$.

+) We consider $S := \{(u, \lambda, x) \in \mathbb{C}_u^3 \times \mathbb{C}_\lambda^2 \times \mathbb{C}_x^3 \mid L_{f,u} = 0\}$, which is an affine manifold. Since $f_i = f_2 = 0$ are the equations in $L_{f,u}(\lambda, x) = 0$,those are sub-manifolds of $\mathbb{C}_u^3 \times \mathbb{C}_\lambda^2 \times X_{\mathbb{C}}$, where $X_{\mathbb{C}} = \mathrm{N}(f_1, f_2)$ is the complex hypersurface.

Let $x \in X_{\mathbb{C}}$ and denote h for the projection of S to $X_{\mathbb{C}}$. Then the fiber $h^{-1}(x)$ over x is

$\{(u, \lambda) \in \mathbb{C}_u^3 \times \mathbb{C}_\lambda^2 \mid u - x = \lambda_1.\nabla f_1(x) + \lambda_2.\nabla f_2(x)\}$.

It is easy to see that the fiber $h^{-1}(x)$ is homologous to $\mathbb{C}_\lambda^2$, proving that $S \xrightarrow{h} \mathbb{C}_u^3$ is a $\mathbb{C}^2 - bundle$ and dim S = 3.

+) Considering the projection of S to $\mathbb{C}_u^3$ is dominant. By Sard's theorem, the general fiber has dimension $3 - 3 = 0$ and smooth. It means that when $u \in \mathbb{C}_u^3$ is general, the $L_{f,u}(\lambda, x) = 0$ has finite solutions, i.e. $L_{f,u}(\lambda, x) = 0$ has a finite number of critical points, and the number of critical points is independent of *u*.

+) Let $Y \subset X_{\mathbb{C}}$ be the set of points of $X_{\mathbb{C}}$ that lie on some coordinate plane i.e do not lie in $\left(\mathbb{C}^{\times}\right)^3$. Since $\dim X_{\mathbb{C}} = n - m = 3 - 2 = 1$, Y has dimension 1-1=0 and its inverse image $h^{-1}(Y)$ in S has dimension 0+1 =1.

+) The points $u \in \mathbb{C}_u^3$ that have a solution (x, λ) to $L_{f,u}(\lambda, x) = 0$ with $x \notin \left(\mathbb{C}^{\times}\right)^3$ make up the image *Im* of Y under the projection to $\mathbb{C}_u^3$. Therefore, *Im* has dimension at most 1, this says that all solution to $L_{f,u}(\lambda, x) = 0$ lie in $\left(\mathbb{C}^{\times}\right)^5$ when *u* is general.

Since $L_{f,u}(\lambda, x) = 0$ has finite critical points and lies in $\left(\mathbb{C}^{\times}\right)^5$, on the other hand according to Bernstein's theorem when $m = 5$, the number of solutions of $f_i = f_2 = 0$ in $\left(\mathbb{C}^{\times}\right)^5$ is less than or equal to $MV(Q_1, Q_2, Q_3, Q_4, Q_5)$ so the number of critical points of $L_{f,u}(\lambda, x) = 0 \le MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'})$

or $EDD(f_1, f_2) \le MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'})$,

where P_1, P_2 are the Newton polytopes of f_1, f_2 and $P_{i'}$ is the Newton polytopes of $u - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2$ for $i = 1, 2, 3$.

□

5. Conclusion

Bernstein's Theorem is the foundation of our proof strategy. This result provides an efficient method for demonstrating that the number of solutions to a polynomial equation system can be expressed as a mixed volume. We hope that our research inspires new lines of research that use this approach in applications other than EDD.

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