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On the p-curvatures of t-connections over a relative smooth projective scheme

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Abstract

Let t be a global function of an integral R-scheme S and X be a smooth projective scheme over R. We show that the characteristic polynomial of the p $-$curvature of an integrable t $-$connection over X_S is horizontal.

Keywords: t-bundles, p-curvature, Hitchin morphism.

1. Introduction

The theory of linear differential operators in positive characteristic was initiated in the 1970s by Katz [Kat70, Kat82], Dwork [Dwo82] and Honda [Hon81]. The aim of these works is to connect with the studying local-global principle for linear ordinary differential equations which is known as the p-curvature conjecture of A. Grothendieck in 1969.

Let k be an algebraically closed field of characteristic $char(k) = p > 0$ and let X be a smooth projective curve over k. Let S be a k $-$scheme equipped with a t $-$function. An integrable t-bundle over X_S is a pair (E, ∇_t) containing a vector bundle E over X_S equipped with an integrable t $-$connection ∇_t. The p $-$curvature of (E, ∇_t), denoted by Ψ_{∇_t}, is p $-$linear which defines an element of $\mathcal{H}om_{\mathcal{O}_X}\left(E, E \otimes Fr^* \Omega^1_{\mathcal{O}_X/S}\right)$. It is considered as an $n \times n$ $-$matrix Ψ with coefficients in $\mathcal{O}_{X_S}$ satisfying the following conditions

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$$(1) \qquad \begin{cases} \partial.\det\left(Id - T\Psi_{\nabla_t}\right) = 0; \\ tr\left(\Psi^n.\partial\Psi\right) = 0 \end{cases} \qquad n \geq 0, n \in \mathbb{N},$$

see [LP01, Proposition 3.2].

Let R be a domain of k −algebras and S be an integral R −scheme and X be a smooth projective scheme over R. A Higgs bundle on X_S (relatively over R) is a vector bundle E on X_S equipped with a *Higgs field* $\theta : E \to E \otimes \Omega^1_{X_S/S}$ satisfying $\theta \wedge \theta = 0$, see [S94]. The coefficients of $\det(\lambda - \theta) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n$ belong to $\mathcal{A}_{X_S/S} = \bigoplus H^0\left(X_S, S^i\Omega^1_{X_S/S}\right)$. Let us write $\mathcal{A}_{X/R} = \bigoplus H^0\left(X, S^i\Omega^1_{X/R}\right)$. Denote by $\mathcal{M}_{Dol}(X/R)$ the moduli stack over (Sch/R) which associates to a scheme S the category of finite rank Higgs bundles on $X_S = X \times_R S$. The moduli stack $\mathcal{M}_{Dol}(X/R)$ is equipped with a morphism $h : \mathcal{M}_{Dol}(X/R) \to \mathcal{A}_{X/R}$ which is defined by sending each Higgs bundle (E, θ) to the coefficients of the characteristic polynomial of \theta. The morphism h is called the *Hitchin morphism* of X, see e.g. [Hit87, S94].

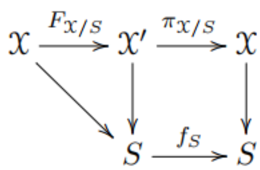
Denote by $\mathcal{C}(n, X/k)$ for the moduli stack of integrable t-bundle over X_S. The characteristic polynomial of the p-curvature Ψ_{∇_t} of ∇_t defines a morphism $\underline{Char} : \mathcal{C}(n, X/k) \to \mathcal{A}_{X/k'} \times \mathbb{A}^1_k$. In the case where X is a smooth projective curve over k, by using (1), Laszlo and Pauly showed that there exists a morphism $\mathcal{H} : \mathcal{C}(r, X/k) \to \mathcal{A}_{X/k} \times \mathbb{A}^1_k$ of stacks such that the restriction $\mathcal{H}|_{t=0}$ is the Hitchin morphism of X. In this manuscript, we establish an analog version of formula (1), Theorems 3.1 and 3.5, for the case where X is a smooth projective scheme over R. These formulas then allow us to define the Hitchin morphism

$$\mathcal{H} : \mathcal{C}(r, X/R) \to \mathcal{A}_{X/R} \times \mathbb{A}^1_R.$$

2. Preliminaries

Let R be an algebra over k and let S be an R −scheme endowed with a global function t. Let $\mathfrak{X}$ be a smooth projective R −scheme and denote by $X_S = X \times_R S$. As in [LP01], we also denote by t for the pullback of t to the scheme X_S by $X_S \to X$.

2.1. Frobenius morphisms. Let us denote by $f_S : S \to S$ the absolute Frobenius, which is topologically the identity and the p-power on functions, and S' the inverse image of S by the Frobenius f_S. Let $\mathfrak{X}' = X_S \times_{f_S} S$ be the pullback of $\mathfrak{X}$ by f_S. There is a unique S-morphism $F_{\mathfrak{X}/S} : \mathfrak{X} \to \mathfrak{X}'$ such that the diagram



commutes. The S-morphism $F_{\mathfrak{X}/S}$ is called the *relative Frobenius morphism* of $\mathfrak{X}$ over S.

Moreover, if $S = spec(A)$ the spectrum of an R-algebra A and $\mathfrak{X} \subset \mathbb{A}_A^N$ is given by equations $f_j = \sum_I a_I x^I$ together with coordinates (x_i), where $1 \le i \le N, a_{I,j} \in A$, then $\mathfrak{X}' \subset \mathbb{A}_A^N$ is defined by $f_j^{[p]} = \sum_I a_I^p x^I$.

2.2. Local systems. Let us write $\mathcal{D}er(\mathfrak{X}/S)$ for the sheaf of germs of S-derivatives on $\mathcal{O}_{\mathfrak{X}}$. As $\mathcal{O}_{\mathfrak{X}}$-modules, this sheaf is isomorphic to $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\Omega^1_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}})$.

Definition 2.1. A local system on $\mathfrak{X}$ is a rank n vector bundle E on $\mathfrak{X}$ equipped with an integrable connection $\nabla : E \to E \otimes \Omega^1_{\mathfrak{X}/S}$. A S-connection $\nabla : E \to E \otimes \Omega^1_{\mathfrak{X}/S}$ is called *integrable* if the composition of it with the induced map $\nabla : E \otimes \Omega^1_{/S} \to E \otimes \Omega^2_{\mathfrak{X}/S}$ is zero.

Let (E, ∇) be a local system on $\mathfrak{X}$. By using duality, ∇ gives rise to an $\mathcal{O}_{\mathfrak{X}}$-linear map

$$\nabla : \mathcal{D}er(\mathfrak{X}/S) \to \mathcal{E}nd_S(E)$$

sending each $D \in \mathcal{D}er(\mathfrak{X}/S)$ to $\nabla(D)$ in $\mathcal{E}nd(E)$, where $\nabla(D)$ is the composite

$$E \xrightarrow{\nabla} E \otimes \Omega^1_{\mathfrak{X}/S} \xrightarrow{1\otimes D} E \otimes \mathcal{O}_{\mathfrak{X}}.$$

We also note that, according to [Kat70], the p^{th}-iterate $D^p \in \mathcal{D}er(\mathfrak{X}/S)$ for each $D \in \mathcal{D}er(\mathfrak{X}/S)$. As in [Section 5.0, Kat70], the p-curvature of the connection ∇ is a mapping of sheaves $\Psi_\nabla : \mathcal{D}er(\mathfrak{X}/S) \to \mathcal{E}nd_{\mathfrak{X}}(E)$ by $\Psi_\nabla(D) = \nabla(D)^p - \nabla(D^p)$. It is p-linear, i.e., an additive map and $\Psi_\nabla(fe) = f^p \Psi_\nabla(e)$ for all f and e are local sections of $\mathcal{O}_{\mathfrak{X}}$ and E respectively over an open subset of $\mathfrak{X}$.

2.3. t-connections. Let us review t-connections on a vector bundle.

Definition 2.2. Let E be a vector bundle of rank n over $\mathfrak{X}$ and $\nabla_t : E \to E \otimes \Omega^1_{\mathfrak{X}/S}$ be an $\mathcal{O}_S$-linear map.

a) The map ∇_t is called a *t-connection* on E if $\nabla_t(ae) = tda \otimes e + a\nabla_t(e)$, where a and e are local sections of $\mathcal{O}_{\mathfrak{X}}$ and E respectively over an open subset of $\mathfrak{X}$. We say that the pair (E, ∇_t) is a t-bundle over $\mathfrak{X}$.

b) The t-connection ∇_t is called *integrable* if $\nabla_t \circ \nabla_t = 0$.

Proposition 2.3. Let (E, ∇_t) be a t-connection over $\mathfrak{X}$. Then:

1) The t-connection ∇_t gives rise to an $\mathcal{O}_{\mathfrak{X}}$-linear morphism $\nabla_t : \mathcal{D}er(\mathfrak{X}/S) \to End_S(E)$ sending $D \in \mathcal{D}er(\mathfrak{X}/S)$ to $\nabla_t(D) \in \mathcal{E}nd_S(E)$, where $\nabla_t(D)$ is the composition $E \xrightarrow{\nabla} E \otimes \Omega^1_{\mathfrak{X}/S} \xrightarrow{1\otimes D} E \otimes \mathcal{O}_{\mathfrak{X}}$.

2) The t-connection ∇_t is integrable precisely when $[\nabla_t(D), \nabla_t(D')] = \nabla_t([D, D'])$ for all $D, D' \in \mathcal{D}er(\mathfrak{X}/S)$.

Proof. See [I.0.5, Kat70].

Proposition 2.4. Let (E,∇_t) be a t-bundle on $\mathfrak{X}$ and let s be a section of E respectively over an open subset of $\mathfrak{X}$. Then $\nabla_t(s)=0$ iff $\nabla_t(D)(s)=0$ for all $D\in\mathcal{D}er(\mathfrak{X}/S)$.

Proof. The proof is clear.

2.4. The characteristic polynomial of p-curvatures. Let ∇_t be a t-bundle over $\mathfrak{X}$. According to [Kat70], the map from $\mathcal{D}er(\mathfrak{X}/S)$ to $\mathcal{E}nd_S(E)$ which sends each D to $[\nabla_t(D)]^p - t^{p-1}\nabla_t(D^p)$ is additive since ∇_t is so.

Definition 2.5. The additive morphism

$$\Psi_{\nabla_t}:\mathcal{D}er(\mathfrak{X}/S)\to\mathcal{E}nd_S(E);\quad D\mapsto[\nabla_t(D)]^p - t^{p-1}\nabla_t(D^p)$$

is called the *p-curvature* of (E,∇_t).

Example 2.6. Let E be a rank n vector bundle over $\mathfrak{X}$. An 0-connection $\nabla_0: E\to E\otimes\Omega^1_{\mathfrak{X}/S}$ is a Higgs field on E. Then, its p-curvature $\Psi_{\nabla_0}=[\nabla_0]^p$.

The following is referred to Lemma 3.3 in [LP01].

Proposition 2.7. Let Ψ_{∇_t} be the p-curvature of (E,∇_t). Then

1) The p-curvature of a 0--connection on E has form p-power of a Higgs field.
2) The map Ψ_{∇_t} is p-linear. In particular, it defines an element in $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}\left(E, E\otimes Fr^*\Omega^1_{\mathfrak{X}'/S}\right)$, (we still denote it by Ψ_{∇_t}).

Proof. The assertion (1) is obvious. The proof of (2) in fact is given by adapting Proposition 5.2.0 of [Kat70].

According to Proposition 5.2.1-2 of [Kat70], we obtain that

Lemma 2.8. Let (E,∇_t) be a t-bundle over $\mathfrak{X}$ and assume that ∇_t is integrable. Then $\left[\nabla_t(D),\Psi_{\nabla_t}(D')\right]=\left[\Psi_{\nabla_t}(D),\Psi_{\nabla_t}(D')\right]=0$ for all $D,D'\in\mathcal{D}er(\mathfrak{X}/S)$.

Proof. We first note that ∇_t is integrable and Ψ_{∇_t} is p-linear. Hence, By adapting the proof of Proposition 5.2.1-2 of [Kat70], we obtain that $\left[\nabla_t(D),\Psi_{\nabla_t}(D')\right]=\left[\Psi_{\nabla_t}(D),\Psi_{\nabla_t}(D')\right]=0$ for all $D,D'\in\mathcal{D}er(\mathfrak{X}/S)$.

Let (E,∇_t) be an integrable t-bundle over $\mathfrak{X}$. Then, Ψ_{∇_t} can be seen as a section of $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}\left(E,E\otimes Fr^*\Omega^1_{\mathfrak{X}'/S}\right)$. The polynomial $\det\left(Id - T\Psi_{\nabla_t}\right)$ of Ψ_{∇_t} can be considered as an element in $\bigoplus_{i=1}^n H^0\left(\mathfrak{X}, S^i(Fr^*\Omega^1_{\mathfrak{X}'/S})\right)$. According to Cartier's Theorem [Kat70, Theorem 5.1], there is a canonical connection ∇^{can} on $Fr^*S^i(\Omega^1_{\mathfrak{X}'/S})$ such that the diagram

$$\begin{array}{ccccc} & & Fr^{*}S^{i}\Omega^{1}_{\mathfrak{X}'/S}\otimes\Omega^{1}_{\mathfrak{X}/S} & & \\ & \overset{\nabla^{can}}{\nearrow} & & \overset{Id\otimes D}{\searrow} & \\ Fr^{*}S^{i}\Omega^{1}_{\mathfrak{X}'/S} & & \xrightarrow[\nabla^{can}(D)]{} & & Fr^{*}S^{i}\Omega^{1}_{\mathfrak{X}'/S} \end{array}$$

commutes for each $i=1,\cdots,n$. Similar as in the proof of Proposition 2.4, we can assume that $\mathfrak{X}=spec\left(A[z]\right)$ is affine and fix $(\mathbf{e})=\{e_1,\cdots,e_n\}$ an A[z]-basis of E, where A is an algebra over R. We then write $\det\left(T\Psi_{\nabla_t}-Id\right)=\left(s_1,\cdots,s_n\right)$, where $s_i\in H^0\left(\mathfrak{X},S^i(Fr^*\Omega^1_{\mathfrak{X}'/S})\right)$ for each $i=1,\cdots,n$. Hence, for each $D\in\mathcal{D}er(\mathfrak{X}/S)$, we define $D.\det\left(T\Psi_{\nabla_t}-Id\right):=\left(\nabla^{can}(D)(s_1),\cdots,\nabla^{can}(D)(s_n)\right)$. Note that $\left(\nabla^{can}(D)(s_1),\cdots,\nabla^{can}(D)(s_n)\right)=0$ precisely when $D.\det\left(T\Psi_{\nabla_t}-Id\right)=0$ for all $D\in\mathcal{D}er(\mathfrak{X}/S)$.

Let us write $\Psi^{D}_{\nabla_t}$ for the matrix of $\Psi_{\nabla_t}(D)$ with respect to $(\mathbf{e})$ for each $D\in\mathcal{D}er(\mathfrak{X}/S)$ and write ∂_D for the S-derivative corresponding to $D\in\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_{\mathfrak{X}})$. Since $f^*_{\mathfrak{X}}\mathcal{D}er(\mathfrak{X}/S)=Fr^*\mathcal{D}er(\mathfrak{X}'/S)$, the operator $\partial_{D'}.\det(T\Psi^{D}_{\nabla_t}-Id)$ is well defined for all $D'\in\mathcal{D}er(\mathfrak{X}/S)$.

Proposition 2.9. Let $D\in\mathcal{D}er(\mathfrak{X}/S)$. The following conditions are equivalent:

1) $\partial_D.\det\left(T\Psi^{D'}_{\nabla_t}-Id\right)=0$ for all $D'\in\mathcal{D}er(\mathfrak{X}/S)$.

2) $D.\det\left(T\Psi_{\nabla_t}-Id\right)=0$.

Proof. Without loss of generality, we consider $\mathfrak{X}=spec\left(A[z]\right)$ being affine and fix $(\mathbf{e})$ an A[z]-basis of E, where A is an algebra over R. By using Remark 2.7, the p-curvature Ψ_{∇_t} is presented by the following matrix $Mat(\Psi_{\nabla_t},\mathbf{e})=\left(\sum_{k=1}^{d}a^{ij}_k\otimes_{f_{\mathfrak{X}}}dz_k\right)_{n\times n}$, where $a^{ij}_k\in A[z]$ for each triple (i,j,k) such that $1\le i,j,k\le d$. Firstly, $(2)\Rightarrow(1)$ is obvious. We now prove that $(1)\Rightarrow(2)$. Assume that $\partial_D.\det(T\Psi^{D'}_{\nabla_t}-Id)=0$ for all $D'\in\mathcal{D}er(\mathfrak{X}/S)$. We note that $\Psi^{D'}_{\nabla_t}$ is given by the composition

$$E\xrightarrow{\Psi_{\nabla_t}}E\otimes\Omega^1_{\mathfrak{X}/S}\xrightarrow{1\otimes D'}E\otimes\mathcal{O}_{\mathfrak{X}}.$$

Considering D' corresponds to $\partial_{z_{i_1}}+...+\partial_{z_{i_s}}$, where $1\le i_1<i_2<...<i_s\le d$ and $1\le s\le d$. Because $\partial_D.\det(T\Psi^{D'}_{\nabla_t}-Id)=0$, we obtain that

$$\partial_D.\det\begin{pmatrix} T\sum_{k=1}^{s} a_{i_k}^{11}-1 & T\sum_{k=1}^{s} a_{i_k}^{12} & \dots & T\sum_{k=1}^{s} a_{i_k}^{1n} \\ T\sum_{k=1}^{s} a_{i_k}^{21} & T\sum_{k=1}^{s} a_{i_k}^{22}-1 & \dots & T\sum_{k=1}^{s} a_{i_k}^{2n} \\ \dots & \dots & & \dots \\ T\sum_{k=1}^{s} a_{i_k}^{n1} & T\sum_{k=1}^{s} a_{i_k}^{n2} & \dots & T\sum_{k=1}^{s} a_{i_k}^{nn}-1 \end{pmatrix} = 0.$$

Hence, $D.\det\left(\left(\sum_{k=1}^{d} a_k^{ij} \otimes_{f_{\mathfrak{X}}} dz_k\right)_{n\times n}.T - Id\right) = 0$ which means that $D.\det\left(T\Psi_{\nabla_t} - Id\right) = 0$.

As a direct consequence of Proposition 2.9, we arrive at

Corollary 2.10. Let $P_{\Psi_{\nabla_t}} = (-1)^n T^n + s_1 T^{n-1} + \cdots + s_n$ be the characteristic polynomial of Ψ_{∇_t} and $k \in \mathbb{N}$ such that $1 \le k \le d$. Then $\nabla^{can}(s_i) = 0$ for all $1 \le i \le n$ if and only if $\partial_k.\det(T\Psi_{\nabla_t}^{D} - Id) = 0$ for all $D \in \mathcal{D}er(\mathfrak{X}/S)$.

3. Horizontality of characteristic polynomial of p-curvature

Let $\nabla_t : E \to E \otimes \Omega^1_{X_S/S}$ be an integrable t-connection on E over X_S. By localization, assume that $\mathfrak{X} = spec\left(A[z]\right)$ is affine and fix $(\mathbf{e}) = \{e_1, \cdots, e_n\}$ an A[z]-basis of E, where A is an algebra over R. According to [Mo09], we obtain the following theorem.

Theorem 3.1. The following assertions $\partial_{z_i}.\det(Id - T.\Psi_{\nabla_t}^{\partial_j}) = 0$ hold for all $1 \le i, j \le d$.

Proof. Because we work in a completion of the local ring $\mathcal{O}_{X_S,0}$, the t-connection ∇_t has the coordinate representation $\nabla_t(v) = t\partial_{z_1}(v)dz_1 + \ldots + t\partial_{z_i}(v)dz_d + (A_1 dz_1 + \ldots + A_d dz_d)v,$ where $A_1, \ldots, A_d \in Mat_n(A\ \ z\ )$. Moreover, for each $1 \le i \le d$, we have

$$(2) \qquad \nabla_t(\partial_{z_i}) = t\partial_{z_i} + A_i.$$

Hence, since $(\partial_{z_i})^p = 0$, we immediately obtain that $\Psi_{\nabla_t}^{\partial_{z_i}} = \left(\nabla_t(\partial_{z_i})\right)^p$. On the other hand, the integrability of ∇_t allows us to show that $\left[\nabla_t(\partial_{z_i}), \nabla_t(\partial_{z_j})\right] = 0$ for all $1 \le i, j \le d$. Using Lemma 2.8, we have $[\nabla_t(\partial_{z_i}), \Psi_{\nabla_t}(\partial_{z_j})] = 0$ in $\mathcal{E}nd(E)$ for each $1 \le i, j \le d$. Hence,

$$(3) \qquad \left[\nabla_t(\partial_{z_i}), \Psi_{\nabla_t}^{\partial_{z_j}}\right] = 0; \qquad 1 \le i, j \le d.$$

By putting (2) and (3) together, we obtain that $\left[t\partial_{z_i} + A_i, \Psi_{\nabla_t}^{\partial_j}\right] = 0; 1 \le i, j \le d$. Therefore,

$$(4) \qquad t\left[\partial_{z_i}, \Psi_{\nabla_t}^{\partial_j}\right] = \left[\Psi_{\nabla_t}^{\partial_j}, A_i\right]; \qquad 1 \le i, j \le d.$$

For each $i = 1, \ldots, d$, let us write $\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j}$ for the matrix of partial derivatives of the entries of $\Psi_{\nabla_t}^{\partial_j}$ with respect to z_i. By adapting the proof of Proposition 3.2 in [LP01], we obtain that

(5) $\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j} = \left[\partial_{z_i}, \Psi_{\nabla_t}^{\partial_j}\right]; \qquad 1 \le i, j \le d.$

Now we combine \eqref{VanishDerDeter3} and \eqref{VanishDerDeter5} together, we then obtain that $t.tr\left[(\Psi_{\nabla_t}^{\partial_{z_j}})^n \partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_j}}\right] = tr\left[(\Psi_{\nabla_t}^{\partial_{z_j}})^n [\Psi_{\nabla_t}^{\partial_j}, A_i]\right] = 0,$ for all $n \ge 0$ and $1 \le i, j \le d$. In order to prove $tr\left[(\Psi_{\nabla_t}^{\partial_{z_k}})^n \partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_k}}\right] = 0$, let us consider $t \in A$ as follows:

- Case 1: t=0. By the definition of Ψ_{∇_t}, we have $\Psi_0(\nabla_0)^{\partial_{z_1}} = A_1^p; ...; \Psi_0(\nabla_0)^{\partial_{z_d}} = A_d^p$. This implies that $\left(\Psi_0(\nabla_0)^{\partial_{z_j}}\right)^n \partial_{z_i}.\Psi_0(\nabla_0)^{\partial_{z_j}} = A_j^{pn} \partial_{z_i}.A_j^p$, and hence,

$$tr\left[(\Psi_0(\nabla_0)^{\partial_{z_j}})^n \partial_{z_i}.?_0(\nabla_0)^{\partial_{z_j}}\right] = tr\left[A_j^{pn} \partial_{z_i}.A_j^p\right] = 0$$

for all $n \ge 0$ and $1 \le i, j \le d$. Therefore, $tr\left[(\Psi_0(\nabla_0)^{\partial_{z_k}})^n \partial_{z_i}.\Psi_0(\nabla_0)^{\partial_{z_k}}\right] = 0$ for all $n \ge 0$ and $1 \le i, j \le d$.

- Case 2: $t \ne 0$. Since A is an integral domain, we get $tr\left[(\Psi_{\nabla_t}^{\partial_{z_j}})^n \partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_j}}\right] = 0,$ for all $n \ge 0$ and $1 \le i, j \le d$.

Since the matrix $\Psi_{\nabla_t}^{\partial_{z_j}}$ belongs to $Mat_n\left(R \quad z\right)$, we obtain the Jacobi identity formula as follows (by using induction on d) $\partial_{z_i}.\det\left(Id - T.\Psi_{\nabla_t}^{\partial_k}\right) = -T \det\left(Id - T.\Psi_{\nabla_t}^{\partial_k}\right) \sum_{n \ge 0} T^n tr\left[(\Psi_{\nabla_t}^{\partial_k})^n \partial_{z_i}.\Psi_{\nabla_t}^{\partial_k}\right]$ for all $1 \le i, j \le d$. Therefore, $\partial_{z_i}.\det\left(Id - T.\Psi_{\nabla_t}^{\partial_j}\right) = 0$.

In general case, we first need the following lemma.

Lemma 3.2. Let $M_1, ..., M_d, B \in Mat_n\left(R \quad z\right)$ such that $\left[M_i, M_j\right] = 0$ for all $1 \le i, j \le d$. Then

$$tr\left(M_1^{n_1} ... M_d^{n_d} [M_k, B]\right) = 0$$

for all $1 \le k \le d$ and $(n_1, ..., n_d) \in \mathbb{N}^d$.

Proof of Lemma. Since $tr\left[M_k, B\right] = 0$, so $tr\left(M_1^{n_1} ... M_d^{n_d} [M_k, B]\right) = 0$.

Now, we apply the idea in the proof of Proposition 3.2 in [LP01] to obtain the following.

Proposition 3.3. Let $D \in \mathcal{D}er(X_S / S)$ and assume that $\Psi_{\nabla_t}^D$ and $\partial_{z_i}.\Psi_{\nabla_t}^D$ as above. Then, we have

$$tr\left[(\Psi_{\nabla_t}^D)^m \partial_{z_i}.\Psi_{\nabla_t}^D\right] = 0.$$

Since Ψ_{∇_t} is p-linear, so $\Psi_{\nabla_t}^D = a_1^p \Psi_{\nabla_t}^{\partial_{z_1}} + ... + a_d^p \Psi_{\nabla_t}^{\partial_{z_d}}$. Hence, the integrability of ∇_t implies that the matrix $\partial_{z_i}.\Psi_{\nabla_t}^D$ is

$$\partial_{z_i}.\Psi_{\nabla_t}^D = \partial_{z_i}.\left(a_1^p \Psi_{\nabla_t}^{\partial_{z_1}} + ... + a_d^p \Psi_{\nabla_t}^{\partial_{z_d}}\right) = a_1^p \partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_1}} + ... + a_d^p \partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_d}}.$$

Because $[\Psi_{\nabla_t}^{\partial_{z_i}}, \Psi_{\nabla_t}^{\partial_{z_j}}] = 0$ for all 1\le i,j\le d, we obtain the following decomposition

$$\left[\Psi_{\nabla_t}^{D}\right]^m = \left[a_1^p\Psi_{\nabla_t}^{\partial_{z_1}} + \ldots + a_d^p\Psi_{\nabla_t}^{\partial_{z_d}}\right]^m = \sum_{k_1+\ldots+k_d=m}\binom{m}{k_1,k_2,\ldots,k_d}\prod_{i=1}^{d}\left[a_i^{pk_i}(\Psi_{\nabla_t}^{\partial_{z_i}})^{k_i}\right].$$

where $(k_1,\ldots,k_d)\in\mathbb{N}^d$ such that $k_1+\ldots+k_d=m$ and $\binom{m}{k_1,k_2,\ldots,k_d}=\frac{m!}{k_1!k_2!\cdots k_d!}$. Hence, we have

$$tr\left[(\Psi_{\nabla_t}^{D})^m\partial_{z_i}.\Psi_{\nabla_t}^{D}\right]=\sum_{k=1}^{d}\sum_{k_1+\ldots+k_d=m}\binom{m}{k_1,\ldots,k_d}a_k^p tr\Big(\Big[\prod_{j=1}^{d}a_j^{pk_j}(\Psi_{\nabla_t}^{\partial_{z_j}})^{k_j}\Big]\partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_k}}\Big).$$

We now need the following lemma.

Lemma 3.4. Let $(n_1,\ldots,n_d)\in\mathbb{N}^d$, assume that $\Psi_{\nabla_t}^{\partial_{z_1}},\cdots,\Psi_{\nabla_t}^{\partial_{z_d}}$ and $\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j}$ as above. Then

$$tr\left[(\Psi_{\nabla_t}^{\partial_{z_1}})^{n_1}\ldots(\Psi_{\nabla_t}^{\partial_{z_d}})^{n_d}(\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j})\right]=0$$

for all $1\le i,j\le d$.

Proof of Lemma 3.4. According to Theorem 3.1, the followings

$$t.tr\left[(\Psi_{\nabla_t}^{\partial_{z_1}})^{n_1}\ldots(\Psi_{\nabla_t}^{\partial_{z_d}})^{n_d}(\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j})\right]=tr\Big((\Psi_{\nabla_t}^{\partial_{z_1}})^{n_1}\ldots(\Psi_{\nabla_t}^{\partial_{z_d}})^{n_d}[\Psi_{\nabla_t}^{\partial_j},A_i]\Big)$$

for all $(n_1,\ldots,n_d)\in\mathbb{N}^d$ and $1\le i,j\le d$. Using Lemma 3.2, we get

$$t.tr\left[(\Psi_{\nabla_t}^{\partial_{z_1}})^{n_1}\ldots(\Psi_{\nabla_t}^{\partial_{z_d}})^{n_d}(\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j})\right]=0.$$

We now again use the method in the proof of Theorem \ref{mainThrm01} to obtain

$$tr\left[(\Psi_{\nabla_t}^{\partial_{z_1}})^{n_1}\ldots(\Psi_{\nabla_t}^{\partial_{z_d}})^{n_d}(\partial_{z_i}.\Psi_{\nabla_t}^{\partial_j})\right]=0.$$

Let us obverse from Theorem 3.1 that $tr\Big(\Big(\prod_{j=1}^{d}a_j^{pk_j}(\Psi_{\nabla_t}^{\partial_{z_j}})^{k_j}\Big)\partial_{z_i}.\Psi_{\nabla_t}^{\partial_{z_k}}\Big)=0$ for each $k_1,\ldots,k_d\in\mathbb{N}$. Therefore, we can conclude that $tr\left[(\Psi_{\nabla_t}^{D})^m\partial_{z_i}.\Psi_{\nabla_t}^{D}\right]=0$ for all $m\in\mathbb{N}$.

We now arrive at

Theorem 3.5. For each $D,D'\in\mathcal{D}er(X_S/S)$, we have $D.\det\big(T.\Psi_{\nabla_t}^{D'}-Id\big)=0$

Proof. Let us write $D=\sum_{j=1}^{d}a_j\partial_{z_j}$ with $a_1,\ldots,a_d\in A[z]$. Then, for each $1\le i\le d$, we have the following identity

$$\partial_{z_i}.\det\big(Id-T.\Psi_{\nabla_t}^{D'}\big)=-T\det\big(Id-T.\Psi_{\nabla_t}^{D'}\big)\sum_{m\ge0}T^m tr\left[(\Psi_{\nabla_t}^{D'})^m\partial_{z_i}.\Psi_{\nabla_t}^{D'}\right].$$

Hence, by using Theorem 3.3 we can see that $\partial_{z_i}.\det\big(Id-T.\Psi_{\nabla_t}^{D'}\big)=0$ for all $1\le i\le d$ and $D'\in\mathcal{D}er(X_S/S)$.

Let now us give an application of Theorem 3.5. Denote by $\mathcal{C}(r,X/R)$ the fibered category over $(\mathbf{Aff}/\mathbb{A}_R^1)$ which associates each $t:S\to\mathbb{A}_R^1$ to the category whose objects are pairs (E,∇_t)

containing a degree zero vector bundle E of rank n on X_S equipped with an integrable t-connection ∇_t, and morphisms are isomorphisms which commute with the t-connections (see Section 3.3 in [LP01]). The affine schemes $\mathcal{A}_{X/R}$ and $\mathcal{A}'_{X/R}$ are defined by

$$\mathcal{A}_{X/R}(S) = \mathcal{A}_{X/R} \times_R S; \qquad \mathcal{A}_{X/R'}(S) = \mathcal{A}_{X/R'} \times_R S$$

for any R-scheme S.

Definition 3.6. Let (E, ∇_t) be a t-bundle over X_S. We denote by $Char(\nabla_t)$ the point of $\mathcal{A}'_{X/R}(S)$ which is defined by the coefficients of the characteristic polynomial of the morphism $\Psi_{\nabla_t} : E \to E \otimes q^* S^p \Omega^1_{X/R}$, where $q : X_S \to X$ is the first projection.

According to [LP01], the functor $\nabla_t \mapsto (Char(\nabla_t), t)$ defines a morphism of $\mathbb{A}^1_R$-stacks:

$$\underline{Char} : \mathcal{C}(r, X/R) \to \mathcal{A}_{X/R'} \times \mathbb{A}^1_R$$

over the category $(\mathbf{Aff} / \mathbb{A}^1_R)$. The absolute Frobenius morphism f_X induces an injective p-linear morphism $f^*_{X_S} : \mathcal{A}_{X/R}(S) \to \mathcal{A}_{X/R'}(S)$ for each affine R-scheme S. Therefore, there is a canonical inclusion of $\mathbb{A}^1_T$-stacks (still denoted by f^*_X): $f^*_X : \mathcal{A}_{X/R} \times \mathbb{A}^1_R \to \mathcal{A}_{X/R'} \times \mathbb{A}^1_R$.

Let now $(E, \nabla_t) \in \mathcal{C}(r, X/R)(S)$ be a t-bundle of rank r over X_S. The locality allows us to assume that $X_S = spec(A[z])$ is affine and fix $(\mathbf{e}) = \{e_1, \cdots, e_n\}$ an A[z]-basis of E, where A is an algebra over R, and the characteristic polynomial $\det(T\Psi_{\nabla_t} - Id)$ is $\det(T\Psi_{\nabla_t} - Id) = (s_1, \cdots, s_n)$, where $s_i \in H^0(X_S, S^i(Fr^*\Omega^1_{X_{S'}/S}))$ for each $i = 1, \cdots, n$. By putting Corollary 2.10 and Theorem 3.5 together, we conclude that $\nabla^{can}_{X_S}(s_i) = 0$, where $\nabla^{can}_{X_S}$ is the canonical connection on $Fr^*(q^* S^i \Omega^1_{X'/R})$. This implies that $Char(\nabla_t)$ belongs to the image of the embedding

$$f^*_{X_S} : \mathcal{A}_{X/R}(S) \times \mathbb{A}^1_R \to \mathcal{A}_{X/R'}(S) \times \mathbb{A}^1_R$$

By using Cartier's theorem, there is another way to obtain the following result (see Section 2 in [EG18]) in the version of t-connections.

Proposition 3.7. There exists a unique morphism $\mathcal{H} : \mathcal{C}(r, X/R) \to \mathcal{A}_{X/R} \times \mathbb{A}^1_R$ over $\mathbb{A}^1_R$ satisfying the followings:

a) the diagram

$$\begin{array}{ccc} & \mathcal{A}_{X/R} \times \mathbb{A}^1_R & \\ {\scriptstyle \exists ! \mathcal{H}} \nearrow & & \searrow {\scriptstyle f^*_X} \\ \mathcal{C}(r, X/R) & \xrightarrow{\underline{Char}} & \mathcal{A}'_{X/R} \times \mathbb{A}^1_R \end{array}$$

is commutative;

b) the restriction \mathcal{H}|_{t=0} gives the Hitchin morphism of X.

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