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Necessary optimality conditions for approximate Pareto efficient solutions of nonsmooth fractional interval-valued multiobjective optimization problems

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Abstract

This paper deals with approximate Pareto efficient solutions of a nonsmooth fractional interval-valued multiobjective optimization. We first introduce some types of approximate Pareto efficient solutions of the considered problem by considering the lower-upper interval order relation. Then we apply some advanced tools of variational analysis and generalized differentiation to establish necessary optimality conditions of Karush–Kuhn–Tucker (KKT)-type for these solutions.

Keywords: Fractional interval-valued multiobjective optimization, KKT optimality conditions, Limiting/Mordukhovich subdifferential, Approximate Pareto solutions;

1. Introduction

In this paper, we are interested in approximate solutions of the following fractional multiobjective problem with multiple interval-valued objective functions:

$$LU-\text{Min } F(x) := \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right)$$
$$\text{s. t. } x \in \Omega := \{ x \in S : h_j(x) \le 0, j = 1, \ldots, p \}, \tag{FIMP}$$

where $f_i, g_i : \mathbb{R}^n \to \mathcal{K}_c$, $i \in I := \{1, \ldots, m\}$, are interval-valued functions defined respectively by $f_i(x) = \left[f_i^L(x), f_i^U(x) \right]$, $g_i(x) = \left[g_i^L(x), g_i^U(x) \right]$, f_i^L, f_i^U, g_i^L, $g_i^U : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions satisfying $f_i^L(x) \le f_i^U(x)$ and

$0 < g_i^L(x) \le g_i^U(x),$

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for all $x \in S$ and $i \in I$, $\mathcal{K}_c$ is the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$\mathcal{K}_c = \{[a^L, a^U]:\ a^L, a^U \in \mathbb{R},\ a^L \le a^U\}$,

$h_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J := \{1, \ldots, p\}$, are locally Lipschitz functions, and S is a nonempty and closed subset of $\mathbb{R}^n$.

An interval-valued optimization problem is one of the deterministic optimization models to deal with the uncertain/incomplete data. Over the recent years, there has been growing interest among the researchers to study optimality conditions for interval-valued multiobjective optimization problems; see e.g., [1, 6, 8, 13–15, 17–22].

In contrast with interval-valued multiobjective optimization problems, there are few recent publications devoted to optimality conditions for fractional interval-valued multiobjective optimization problems; see [2–4]. Fractional interval-valued multiobjective optimization problems occur frequently in public policy decision making such as management science, transportation management, education management, medicine, etc; see e.g., [16]. These problems are neither linear nor convex.

To the best of our knowledge, so far there have been no papers investigating optimality conditions for approximate Pareto efficient solutions of fractional interval-valued multiobjective optimization problems with locally Lipschitz data.

Motivated by the above observations, in this paper, we introduce some kinds of approximate Pareto efficient solutions with respect to lower-upper (LU) interval order relation for problems of the form (FIMP). Then we employ the limiting/Mordukhovich subdifferential and the limiting/Mordukhovich normal cone to derive necessary optimality conditions in fuzzy form for these Pareto solutions of this problem.

The paper is organized as follows. Section 2 contains some basic definitions from variational analysis, interval analysis and several auxiliary results. In Section 3, we first introduce some kinds of approximate Pareto efficient solutions of the problem (FIMP) and then establish necessary conditions of KKT-type for these solutions. Section 4 draws some conclusions.

2. Preliminaries

We use the following notation and terminology. Fix $n \in \mathbb{N} := \{1, 2, \ldots\}$. The space $\mathbb{R}^n$ is equipped with the usual scalar product and Euclidean norm. The closed unit ball of $\mathbb{R}^n$ is denoted by $\mathbb{B}_{\mathbb{R}^n}$. We denote the nonnegative orthant in $\mathbb{R}^n$ by $\mathbb{R}^n_+$. The topological closure of S is denoted by $\mathrm{cl}S$.

Definition 2.1. (see [9,10]). Given $\bar{x} \in \mathrm{cl}S$. The set

$$N(\bar{x}; S) := \left\{ z^* \in \mathbb{R}^n : \exists x^k \xrightarrow{S} \bar{x}, \varepsilon_k \to 0^+, z_k^* \to z^*, z_k^* \in N_{\varepsilon_k}(x^k; S),\ \forall k \in \mathbb{N} \right\},$$

is called the *Mordukhovich/limiting normal cone* of S at $\bar{x}$, where

$$N_\varepsilon(x; S) := \left\{ z^* \in \mathbb{R}^n : \limsup_{u \xrightarrow{S} x} \frac{\langle z^*, u - x\rangle}{\| u - x \|} \le \varepsilon \right\},$$

is the set of ε-*normals* of S at x and $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an *extended-real-valued function*. The *epigraph* and *domain* of φ are denoted, respectively, by

$$\text{epi } \varphi := \left\{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \ge \varphi(x)\right\},$$

$$\text{dom } \varphi := \left\{x \in \mathbb{R}^n : |\varphi(x)| < +\infty\right\}.$$

Definition 2.2. (see [9,10]). Let $\bar{x} \in \text{dom } \varphi$.

(i) The set

$$\partial\varphi(\bar{x}) := \left\{x^* \in \mathbb{R}^n : (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\right\},$$

is called the *Mordukhovich/limiting subdifferential* of φ at $\bar{x}$. If $\bar{x} \notin \text{dom } \varphi$, then we put $\partial\varphi(\bar{x}) = \emptyset$.

(ii) The set $\partial^+\varphi(\bar{x}) := -\partial(-\varphi)(\bar{x})$ is called the *upper subdifferential* of φ at $\bar{x}$.

We now summarize some properties of the Mordukhovich subdifferential that will be used in the next section.

Proposition 2.3. (see [9,10]). *Let $\varphi_l : \mathbb{R}^n \to \overline{\mathbb{R}}$, $l = 1,\ldots,p$, $p \ge 2$, be lower semicontinuous around $\bar{x}$ and let all but one of these functions be locally Lipschitz around $\bar{x}$. Then we have the following inclusion:*

$$\partial(\varphi_1 + \ldots + \varphi_p)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \ldots + \partial\varphi_p(\bar{x}).$$

Proposition 2.4. (see [9,10]). *Let $\varphi_l : \mathbb{R}^n \to \overline{\mathbb{R}}$, $l = 1,\ldots,p$, be locally Lipschitz around $\bar{x}$. Then the function $\phi(\cdot) := \max\{\varphi_l(\cdot) : l = 1,\ldots,p\}$ is also locally Lipschitz around $\bar{x}$ and we have*

$$\partial\phi(\bar{x}) \subset \bigcup\left\{\partial\Big(\sum_{l=1}^{p}\lambda_l\varphi_l\Big)(\bar{x}) : (\lambda_1,\ldots,\lambda_p) \in \Lambda(\bar{x})\right\},$$

where $\Lambda(\bar{x}) := \left\{(\lambda_1,\ldots,\lambda_p) : \lambda_l \ge 0, \sum_{l=1}^{p}\lambda_l = 1, \lambda_l[\varphi_l(\bar{x}) - \phi(\bar{x})] = 0\right\}.$

Proposition 2.5. (see [9,10]). *Let $\varphi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,2$, be Lipschitz continuous around $\bar{x}$. If $\varphi_2(\bar{x}) \ne 0$, then we have*

$$\partial\left(\frac{\varphi_1}{\varphi_2}\right)(\bar{x}) \subset \frac{\partial(\varphi_2(\bar{x})\varphi_1)(\bar{x}) + \partial(-\varphi_1(\bar{x})\varphi_2)(\bar{x})}{[\varphi_2(\bar{x})]^2}.$$

Proposition 2.6. (see [9,10]). *Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of φ, then* $0 \in \partial\varphi(\bar{x})$.

Next, we recall the Ekeland variational principle, which is needed for our investigation.

Proposition 2.7. (see [5]). *Let (X, d) be a complete metric space and $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function bounded from below. Let $\varepsilon > 0$ and $x_0 \in X$ be given such that $\varphi(x_0) \le \inf_{x\in X}\varphi(x) + \varepsilon$. Then for any $\lambda > 0$ there exists $\bar{x} \in X$ satisfying the following conditions:*

(i) $\varphi(\bar{x}) \le \varphi(x_0)$,

(ii) $d(\bar{x}, x_0) \le \lambda$,

(iii) $\varphi(\bar{x}) < \varphi(x) + \dfrac{\varepsilon}{\lambda}d(x, \bar{x})$ *for all* $x \in X \setminus \{\bar{x}\}$.

Finally, in this section, we recall some defnitions and facts in interval analysis, see e.g., [1, 11,12]. Let $A=[a^L,a^U]$ and $B=[b^L,b^U]$ be two intervals in $\mathcal{K}_c$. Then, we define

(i) $A+B:=\{a+b\,:\,a\in A, b\in B\}=[a^L+b^L, a^U+b^U];$

(ii) $A-B:=\{a-b: a\in A, b\in B\}=[a^L-b^U, a^U-b^L];$

(iii) $kA:=\{ka\,:\,a\in A\}=\begin{cases}[ka^L,ka^U] \text{ if } k\geq 0,\\ [ka^U,ka^L] \text{ if } k<0;\end{cases}$

(iv) $\dfrac{A}{B}:=\left[\min\left(\dfrac{a^L}{b^L},\dfrac{a^L}{b^U},\dfrac{a^U}{b^L},\dfrac{a^U}{b^U}\right),\max\left(\dfrac{a^L}{b^L},\dfrac{a^L}{b^U},\dfrac{a^U}{b^L},\dfrac{a^U}{b^U}\right)\right]$ if $0\notin B$.

Definition 2.7. Let $A=[a^L,a^U]$ and $B=[b^L,b^U]$ be two intervals in $\mathcal{K}_c$. We say that:

(i) $A\leq_{LU} B$ if $a^L\leq b^L$ and $a^U\leq b^U$.

(ii) $A<_{LU} B$ if $A\leq_{LU} B$ and $A\neq B$, or, equivalently, $A<_{LU} B$ if

$$\begin{cases}a^L<b^L\\ a^U\leq b^U,\end{cases} \text{ or } \begin{cases}a^L\leq b^L\\ a^U<b^U,\end{cases} \text{ or } \begin{cases}a^L<b^L\\ a^U<b^U.\end{cases}$$

(iii) $A<^s_{LU} B$ if $a^L<b^L$ and $a^U<b^U$.

3. Optimality conditions for approximate quasi Pareto efficient solutions

We now introduce approximate solutions of (FIMP) with respect to LU interval order relation. For the sake of convenience, *we always assume hereafter* that $f_i^L(x)\geq 0$, $\forall x\in S$ and $i\in I$. Let ϵ_i^L, ϵ_i^U, $i\in I$, be real numbers satisfying $0\leq\epsilon_i^L\leq\epsilon_i^U$ for all $i\in I$ and put $\mathcal{E}:=(\mathcal{E}_1,\ldots,\mathcal{E}_m)$, where $\mathcal{E}_i:=[\epsilon_i^L,\epsilon_i^U]$. For each $i\in I$ and $x\in\mathbb{R}^n$, put $F_i(x):=\dfrac{f_i(x)}{g_i(x)}$. By definition, we have

$$F_i(x):=\frac{f_i(x)}{g_i(x)}=\left[\frac{f_i^L(x)}{g_i^U(x)},\frac{f_i^U(x)}{g_i^L(x)}\right].$$

Definition 3.1. Let $\bar{x}\in\Omega$. We say that:

(i) $\bar{x}$ is a *type-1* $\mathcal{E}$*-Pareto solution* of (FIMP), denoted by $\bar{x}\in\mathcal{E}-\mathcal{S}_1(\text{FIMP})$, if there is no $x\in\Omega$ such that

$$\begin{cases}F_i(x)\leq_{LU} F_i(\bar{x})-\mathcal{E}_i, & \forall i\in I,\\ F_k(x)<_{LU} F_k(\bar{x})-\mathcal{E}_k, & \text{for at least one } k\in I.\end{cases}$$

(ii) $\bar{x}$ is a *type-2* $\mathcal{E}$*-Pareto solution* of (FIMP), denoted by $\bar{x}\in\mathcal{E}-\mathcal{S}_2(\text{FIMP})$, if there is no $x\in\Omega$ such that

$$\begin{cases}F_i(x)\leq_{LU} F_i(\bar{x})-\mathcal{E}_i, & \forall i\in I,\\ F_k(x)<^s_{LU} F_k(\bar{x})-\mathcal{E}_k, & \text{for at least one } k\in I.\end{cases}$$

(iii) $\bar{x}$ is a *type-1* $\mathcal{E}$*-weakly Pareto solution* of (FIMP), denoted by $\bar{x}\in\mathcal{E}-\mathcal{S}_1^w(\text{FIMP})$, if there is no $x\in\Omega$ such that

$$F_i(x)<_{LU} F_i(\bar{x})-\mathcal{E}_i,\quad \forall i\in I.$$

(iv) $\bar{x}$ is a *type-2* $\mathcal{E}$*-weakly Pareto solution* of (FIMP), denoted by $\bar{x}\in\mathcal{E}-\mathcal{S}_2^w(\text{FIMP})$, if there is no $x\in\Omega$ such that

$F_i(x) <_{LU}^s F_i(\overline{x}) - \mathcal{E}_i, \ \ \forall i \in I.$

By definition, it is easy to see that the following inclusions holds:

(i) $\mathcal{E} - \mathcal{S}_1(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_2(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_2^w(\text{FIMP})$;

(ii) $\mathcal{E} - \mathcal{S}_1(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_1^w(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_2^w(\text{FIMP})$.

Definition 3.2. Let $\overline{x} \in \Omega$. We say that:

(i) $\overline{x}$ is a *type-1 $\mathcal{E}$-quasi Pareto solution* of (FIMP), denoted by $\overline{x} \in \mathcal{E} - \mathcal{S}_1^q(\text{FIMP})$, if there is no $x \in \Omega$ such that

$$\begin{cases} F_i(x) \leq_{LU} F_i(\overline{x}) - \mathcal{E}_i \|x - \overline{x}\|, & \forall i \in I, \\ F_k(x) <_{LU} F_k(\overline{x}) - \mathcal{E}_k \|x - \overline{x}\|, & \text{for at least one } k \in I. \end{cases}$$

(ii) $\overline{x}$ is a *type-2 $\mathcal{E}$-quasi Pareto solution* of (FIMP), denoted by $\overline{x} \in \mathcal{E} - \mathcal{S}_2^q(\text{FIMP})$, if there is no $x \in \Omega$ such that

$$\begin{cases} F_i(x) \leq_{LU} F_i(\overline{x}) - \mathcal{E}_i \|x - \overline{x}\|, & \forall i \in I, \\ F_k(x) <_{LU}^s F_k(\overline{x}) - \mathcal{E}_k \|x - \overline{x}\|, & \text{for at least one } k \in I. \end{cases}$$

(iii) $\overline{x}$ is a *type-1 $\mathcal{E}$-quasi weakly Pareto solution* of (FIMP), denoted by $\overline{x} \in \mathcal{E} - \mathcal{S}_1^{qw}(\text{FIMP})$, if there is no $x \in \Omega$ such that

$F_i(x) <_{LU} F_i(\overline{x}) - \mathcal{E}_i \|x - \overline{x}\|, \ \forall i \in I.$

(iv) $\overline{x}$ is a *type-2 $\mathcal{E}$-quasi weakly Pareto solution* of (FIMP), denoted by $\overline{x} \in \mathcal{E} - \mathcal{S}_2^{qw}(\text{FIMP})$, if there is no $x \in \Omega$ such that

$F_i(x) <_{LU}^s F_i(\overline{x}) - \mathcal{E}_i \|x - \overline{x}\|, \ \forall i \in I.$

We note here that, when $\mathcal{E} = 0$, i.e., $\epsilon_i^L = \epsilon_i^U = 0$, $i \in I$, the notion of a type-1 $\mathcal{E}$-(quasi) Pareto solution (resp., a type-2 $\mathcal{E}$-(quasi) Pareto solution, a type-1 $\mathcal{E}$-(quasi) weakly Pareto solution, a type-2 $\mathcal{E}$-(quasi) weakly Pareto solution) defined above coincides with the one of a *type-1 Pareto solution* (resp., a *type-2 Pareto solution*, a *type-1 weakly Pareto solution*, a *type-2 weakly Pareto solution*); see, e.g., [17,18,21,22]. Furthermore, by definition, it is easy to see that the following inclusions holds:

(i) $\mathcal{E} - \mathcal{S}_1^q(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_2^q(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_2^{qw}(\text{FIMP})$;

(ii) $\mathcal{E} - \mathcal{S}_1^q(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_1^{qw}(\text{FIMP}) \subset \mathcal{E} - \mathcal{S}_2^{qw}(\text{FIMP})$.

In order to obtain necessary optimality conditions of KKT-type for approximate quasi Pareto solutions of (FIMP), we need the following constraint qualification condition, see [9,10].

Definition 3.3. Let $\overline{x} \in \Omega$. We say that:

(i) The *constraint qualification* (CQ) is satisfied at x if

$$N(x;\Omega) \subset \bigcup_{\mu \in A(x)} \sum_{j \in J} \mu_j \partial h_j(x) + N(x;S), \tag{3.1}$$

where $A(x) := \{\mu \in \mathbb{R}_+^p : \mu_j h_j(x) = 0\}$.

(ii) The *robust constraint qualification* (RCQ) is satisfied at $\overline{x}$ if there exists a $\delta > 0$ such that (3.1) holds for all $x \in B(\overline{x}, \delta)$.

The following result provides a KKT-type necessary optimality condition in a fuzzy form for type-

1(type 2) $\mathcal{E}$ -(weakly) Pareto solutions of problem (FIMP).

Theorem 3.4. *Let* $\bar{x} \in \mathcal{E} - \mathcal{S}_2^w(\text{FIMP})$. *If the* (*RCQ*) *holds at* $\bar{x}$ *, then, for any* $\delta > 0$ *small enough, there exist* $x_\delta \in \Omega,\ \beta_i^L \geq 0,\ \beta_i^U \geq 0,\ i \in I$ *, and* $\mu \in A(x_\delta)$ *with* $\sum_{i\in I}(\beta_i^L + \beta_i^U) = 1,$ *such that* $\|x_\delta - \bar{x}\| < \delta$ *and*

$$0 \in \sum_{i\in I} \frac{\beta_i^L}{g_i^U(x_\delta)}\left[\partial f_i^L(x_\delta) - \frac{f_i^L(x_\delta)}{g_i^U(x_\delta)}\partial^+ g_i^U(x_\delta)\right] + \sum_{i\in I}\frac{\beta_i^U}{g_i^L(x_\delta)}\left[\partial f_i^U(x_\delta) - \frac{f_i^U(x_\delta)}{g_i^L(x_\delta)}\partial^+ g_i^L(x_\delta)\right]$$

$$+\frac{1}{\delta}\max_{i\in I, j\in J}\left\{\epsilon_i^U, \epsilon_i^L\right\}\mathbb{B}_{\mathbb{R}^n} + \sum_{j\in J}\mu_j \partial h_j(x_\delta) + N(x_\delta; S), \tag{3.2}$$

$$\beta_i^L\left(\frac{f_i^L(x_\delta)}{g_i^U(x_\delta)} - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} + \epsilon_i^U - \psi(x_\delta)\right) = 0,\ \beta_i^U\left(\frac{f_i^U(x_\delta)}{g_i^L(x_\delta)} - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} + \epsilon_i^L - \psi(x_\delta)\right) = 0.$$

Proof. Since $\bar{x} \in \mathcal{E} - \mathcal{S}_2^w(\text{FIMP})$, there is no $x \in \Omega$ such that $F_i(x) <_{LU}^s F_i(\bar{x}) - \mathcal{E}_i$, $\forall i \in I$, or, equivalently,

$$\frac{f_i^L(x)}{g_i^U(x)} < \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} - \epsilon_i^U \ \text{ and } \ \frac{f_i^U(x)}{g_i^L(x)} < \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} - \epsilon_i^L,\ \forall i \in I.$$

Hence for each $x \in \Omega$, there exists $i \in I$, such that

$$\frac{f_i^L(x)}{g_i^U(x)} \geq \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} - \epsilon_i^U \ \text{ or } \ \frac{f_i^U(x)}{g_i^L(x)} \geq \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} - \epsilon_i^L. \tag{3.3}$$

Let ψ be a real-valued function defined by

$$\psi(x) := \max_{i\in I, j\in J}\left\{\frac{f_i^L(x)}{g_i^U(x)} - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} + \epsilon_i^U, \frac{f_i^U(x)}{g_i^L(x)} - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} + \epsilon_i^L\right\},\ \forall x \in \mathbb{R}^n.$$

By (3.3), we have

$$\psi(x) \geq 0, \quad \forall x \in \Omega.$$

Hence,

$$\psi(\bar{x}) \leq \inf_{x\in\Omega}\psi(x) + \max_{i\in I, j\in J}\left\{\epsilon_i^U, \epsilon_i^L\right\}.$$

By Proposition 2.7, for any $\delta > 0,$ there exists $x_\delta \in \Omega$ such that $\|x_\delta - \bar{x}\| < \delta$ and

$$\psi(x_\delta) \leq \psi(x) + \frac{1}{\delta}\max_{i\in I, j\in J}\left\{\epsilon_i^U, \epsilon_i^L\right\}\|x - x_\delta\| \quad \forall x \in \Omega.$$

Thus x_δ is a minimizer of the function $\varphi(\cdot) := \psi(\cdot) + \frac{1}{\delta}\max_{i\in I, j\in J}\left\{\epsilon_i^U, \epsilon_i^L\right\}\|\cdot - x_\delta\|$ on Ω. This means that x_δ is a minimizer to the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n}\varphi(x) + \sigma(x;\Omega),$$

where $\sigma(\cdot;\Omega)$ is the indicator function of Ω and defined by

$$\sigma(x;\Omega) = \begin{cases} 0, & \text{if } x \in \Omega, \\ +\infty, & \text{otherwise.}\end{cases}$$

By Proposition 2.6, we have

$$0 \in \partial\left(\varphi + \sigma(\cdot;\Omega)\right)(x_\delta).$$

Clearly, φ is locally Lipschitz around x_δ and $\sigma(\cdot;\Omega)$ is lower semicontinuous around this point. Hence by Proposition 2.3 and the fact that $\partial\sigma(\cdot;\Omega)(x_\delta)=N(x_\delta;\Omega)$ (see, e.g., [9, Proposition 1.19]), we obtain

$0\in\partial\varphi(x_\delta)+N(x_\delta;\Omega).$ (3.4)

Applying Proposition 2.3 and the fact that $\partial\left(\|\cdot-x_\delta\|\right)(x_\delta)=\mathbb{B}_{\mathbb{R}^n}$ (see [7, Example 4, p. 198], we get

$$0\in\partial\psi\left(x_\delta\right)+\frac{1}{\delta}\max_{i\in I,\,j\in J}\left\{\epsilon_i^U,\epsilon_i^L\right\}\mathbb{B}_{\mathbb{R}^n}. \tag{3.5}$$

By Proposition 2.4, we have

$$\begin{aligned}\partial\psi(x_\delta)\quad\subset\Bigg\{&\sum_{i\in I}\beta_i^L\partial\left(\frac{f_i^L}{g_i^U}\right)(x_\delta)+\sum_{i\in I}\beta_i^U\partial\left(\frac{f_i^U}{g_i^L}\right)(x_\delta):\beta_i^L,\beta_i^U\ge 0, i\in I,\ \sum_{i\in I}(\beta_i^L+\beta_i^U)=1,\\ &\beta_i^L\left(\frac{f_i^L(x_\delta)}{g_i^U(x_\delta)}-\frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}+\epsilon_i^U-\psi\left(x_\delta\right)\right)=0,\ \beta_i^U\left(\frac{f_i^U(x_\delta)}{g_i^L(x_\delta)}-\frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}+\epsilon_i^L-\psi\left(x_\delta\right)\right)=0\Bigg\}.\end{aligned}\tag{3.6}$$

Now, taking Proposition 2.5 into account, we arrive at

$$\begin{aligned}\partial\left(\frac{f_i^L}{g_i^U}\right)(x_\delta)\quad&\subset\frac{\partial\left(g_i^U(x_\delta)f_i^L\right)(x_\delta)+\partial\left(-f_i^L(x_\delta)g_i^U\right)(x_\delta)}{[g_i^U(x_\delta)]^2}\\ &=\frac{g_i^U(x_\delta)\partial f_i^L(x_\delta)+f_i^L(x_\delta)\partial\left(-g_i^U\right)(x_\delta)}{[g_i^U(x_\delta)]^2},\\ &=\frac{g_i^U(x_\delta)\partial f_i^L(x_\delta)-f_i^L(x_\delta)\partial^+g_i^U(x_\delta)}{[g_i^U(x_\delta)]^2},\ \ \forall i\in I,\end{aligned}\tag{3.7}$$

where the equality holds due to the fact that $f_i^L(x_\delta)\ge 0,\ g_i^U(x_\delta)>0$ and

$\partial\left(-g_i^U\right)\left(x_\delta\right)=-\partial^+g_i^U\left(x_\delta\right),\ \ \forall i\in I.$

Similarly, we have

$$\partial\left(\frac{f_i^U}{g_i^L}\right)(x_\delta)\subset\frac{g_i^L(x_\delta)\partial f_i^U(x_\delta)-f_i^U(x_\delta)\partial^+g_i^L(x_\delta)}{[g_i^L(x_\delta)]^2}\ \ \forall i\in I. \tag{3.8}$$

Since the (RCQ) holds at $\bar{x}$, there is $\bar{\delta}>0$ such that for any $\delta\in\left(0,\bar{\delta}\right)$ there exists $\mu\in A\left(x_\delta\right)$ satisfying

$$N(x_\delta;\Omega)\subset\sum_{j\in J}\mu_j\partial h_j(x_\delta)+N(x_\delta;S). \tag{3.9}$$

To finish the proof of the theorem, it remains to combine (3.4) – (3.9). □

We next establish KKT necessary optimality conditions for type-1(2) $\mathcal{E}$-quasi (weakly) Pareto solutions of problem (FIMP).

Theorem 3.5. *Let* $\bar{x}\in\mathcal{E}-\mathcal{S}_2^{qw}\left(\text{FIMP}\right)$. *If the* (*CQ*) *holds at* $\bar{x}$, *then there exist* $\beta_i^L\ge 0,\ \ \beta_i^U\ge 0,\ i\in I$ *, and* $\mu\in A\left(\bar{x}\right)$ *with* $\sum_{i\in I}(\beta_i^L+\beta_i^U)=1$ *such that*

$$0 \in \sum_{i\in I} \frac{\beta_i^L}{g_i^U(\bar{x})}\left[\partial f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\partial^+ g_i^U(\bar{x})\right] + \sum_{i\in I} \frac{\beta_i^U}{g_i^L(\bar{x})}\left[\partial f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\partial^+ g_i^L(\bar{x})\right]$$
$$+ \sum_{i\in I}(\beta_i^L \epsilon_i^U + \beta_i^U \epsilon_i^L)\, \mathbb{B}_{\mathbb{R}^n} + \sum_{j\in J}\mu_j \partial h_j(\bar{x}) + N(\bar{x}; S).$$

(3.10)

Proof. Since $\bar{x} \in \mathcal{E} - \mathcal{S}_2^{qw}(\text{FIMP})$, there is no $x \in \Omega$ such that $F_i(x) <_{LU}^s F_i(\bar{x}) - \mathcal{E}_i \|x-\bar{x}\|$, $\forall i \in I$, or, equivalently,

$$\frac{f_i^L(x)}{g_i^U(x)} < \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} - \epsilon_i^U \|x-\bar{x}\| \text{ and } \frac{f_i^U(x)}{g_i^L(x)} < \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} - \epsilon_i^L \|x-\bar{x}\|, \ \forall i \in I.$$

Hence for each $x \in \Omega$, there exists $i \in I$, such that

$$\frac{f_i^L(x)}{g_i^U(x)} \geq \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} - \epsilon_i^U \|x-\bar{x}\| \text{ or } \frac{f_i^U(x)}{g_i^L(x)} \geq \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} - \epsilon_i^L \|x-\bar{x}\|. \qquad (3.11)$$

Let ϕ be a real-valued function defined by

$$\phi(x) := \max_{i\in I, j\in J}\left\{\frac{f_i^L(x)}{g_i^U(x)} - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} + \epsilon_i^U \|x-\bar{x}\|, \frac{f_i^U(x)}{g_i^L(x)} - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} + \epsilon_i^L \|x-\bar{x}\|\right\}, \ \forall x \in \mathbb{R}^n.$$

By (3.11), we have

$$0 = \phi(\bar{x}) \leq \phi(x), \ \forall x \in \Omega.$$

This means that $\bar{x}$ is a minimizer to the following unconstrained optimization problem

$$\text{minimizer } \phi(x) + \delta(x;\Omega), \ x \in \mathbb{R}^n.$$

By Proposition 2.6, we have

$$0 \in \partial(\phi + \delta(\cdot;\Omega))(\bar{x}).$$

Hence

$$0 \in \partial\phi(\bar{x}) + N(\bar{x};\Omega).$$

(3.12)

By Propositions 2.4–2.5 and the fact that $\partial(\|\cdot - \bar{x}\|)(\bar{x}) = \mathbb{B}_{\mathbb{R}^n}$, we get

$$\partial\phi(\bar{x}) \subset \left\{\sum_{i\in I}\beta_i^L\left[\partial\left(\frac{f_i^L}{g_i^U}\right)(\bar{x}) + \epsilon_i^U \mathbb{B}_{\mathbb{R}^n}\right] + \sum_{i\in I}\beta_i^U\left[\partial\left(\frac{f_i^U}{g_i^L}\right)(\bar{x}) + \epsilon_i^L \mathbb{B}_{\mathbb{R}^n}\right] : \beta_i^L, \beta_i^U \geq 0, i \in I, \sum_{i\in I}(\beta_i^L + \beta_i^U) = 1\right\}$$
$$\subset \left\{\sum_{i\in I} \frac{\beta_i^L}{g_i^U(\bar{x})}\left[\partial f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\partial^+ g_i^U(\bar{x})\right] + \sum_{i\in I} \frac{\beta_i^U}{g_i^L(\bar{x})}\left[\partial f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\partial^+ g_i^L(\bar{x})\right]\right.$$
$$\left. + \sum_{i\in I}(\beta_i^L \epsilon_i^U + \beta_i^U \epsilon_i^L)\, \mathbb{B}_{\mathbb{R}^n} : \beta_i^L, \beta_i^U \geq 0, i \in I, \sum_{i\in I}(\beta_i^L + \beta_i^U) = 1\right\}.$$

(3.13)

Now, since the (CQ) holds at $\bar{x}$, one has

$$N(\bar{x};\Omega) \subset \bigcup_{\mu\in A(\bar{x})}\sum_{j\in J}\mu_j \partial h_j(\bar{x}) + N(\bar{x}; S). \qquad (3.14)$$

To finish the proof of the theorem, it remains to combine (3.12) – (3.14). □

We close this section by noting that the constraint qualification (CQ) has been widely used in the literature; see e.g., [9,10] for more details. When $S=\mathbb{R}^n$ and $h_j,\ j\in J,$ are continuously differentiable functions at the referenced point, the (CQ) collapses to the well-known Mangasarian-Fromovitz constraint qualification (cf. [10]).

4. Conclusions

In this paper, we discussed about approximate Pareto efficient solutions for a nonsmooth fractional interval-valued multiobjective optimization problem. Eight types of approximate solutions were considered and necessary optimality conditions of KKT-type were derived for these solutions. The obtained results are new. In our further work, we intend to investigate KKT sufficient optimality conditions and duality relations for approximate Pareto efficient solutions of fractional interval-valued multiobjective optimization problems of the form (FIMP).

Declaration of Competing Interest

The authors declare no competing interests.

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