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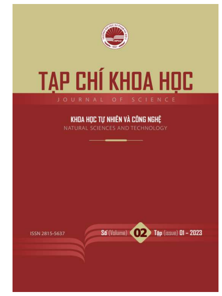


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# An improvement of newton – krylov method for solution of nonlinear equations

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**Abstract**

Solving problems in practice often results in a system of nonlinear equations with a large number of equations and unknowns. Finding the exact solution to this class of equations is very difficult and almost impossible. Recently, with the development of technology, many methods and algorithms have been proposed to approximate the class of these systems of equations. Especially the third-order Newton–Krylov method has solved quite well this class of systems of equations with the third degree of convergence. In this paper, we present a new improvement of the third-order Newton-Krylov method with a quaternary convergence rate and prove the convergence of the iterative formula. In addition, the paper also presents an experimental result to demonstrate the convergence speed of the method.

***Keywords:*** Iterative formula, Convergence, Convergence speed, Nonlinear equations system, Third-order Newton-Krylov method.

## 1. Introduction

Consider a system of nonlinear equations

$$F\left(x\right) = 0\,, \tag{1}$$

where $F = \left(f_1\left(x\right), f_2\left(x\right); \ldots; f_n\left(x\right)\right)^t$ with $f_i : \mathbb{R}^n \to \mathbb{R}$ are nonlinear functions. ( $i = 1,2,\ldots,n$ ).

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Many scientists have researched and proposed methods to solve the system of equations (1) with quadratic convergence, such as Newton's method [1], Chebyshev's method, Halley's method [2] and other methods. The third-order convergent iterative method is presented in [4]-[9]. However, these methods have high computational complexity when the number of equations and unknowns of the system is large.

In 2011, Frontini and Sormani proposed an improved Newton method with a third order convergence rate [8], [9] as follows:

**Newton – Krylov method**

Consider the system of equations

$$F'\left(x_n\right)s_n = -F\left(x_n\right),\ s_n = x_{n+1} - x_n, n \in \mathbb{N}^*. \tag{2}$$

The Newton-Krylov method finds an approximate solution of (2) with condition

$$\left\| F'\left(x_n\right)s_n + F\left(x_n\right) \right\| \le \eta_n \left\| F\left(x_n\right) \right\|,$$

with $\eta_n \in \left[0,1\right)$ is called constraint condition.

**Newton-Krylov aglorithm:**

1. Set $x_0$; $\eta_{max} \in \left[0,1\right)$.
2. Give $n = 0,1,...,$ and:
   - Chose $\eta_n \in \left[0;\eta_{max}\right]$,
   - Apply an iterative method to find $s_n$ of $F'\left(x_n\right)s_n = -F\left(x_n\right)$.

The process will stop if the following condition is satisfied $\left\| F'\left(x_n\right)s_n + F\left(x_n\right) \right\| \le \eta_n \left\| F\left(x_n\right) \right\|$.

- Correct $x_{n+1} = x_n + s_n$.

**Third-order Newton-Krylov method**.

We consider

$$x_{n+1} = x_n - \frac{F\left(x_n\right)}{F'\left(x_n - \frac{1}{2}F'\left(x_n\right)^{-1}F\left(x_n\right)\right)}. \tag{3}$$

To obtain The Newton-Krylov algorithm, we rewrite fomula (3) as follows

$$F'\left(x_n - \frac{1}{2}F'\left(x_n\right)^{-1}F\left(x_n\right)\right)\left(x_{n+1} - x_n\right) = -F\left(x_n\right) \tag{4}$$

Set

$$k\left(x_n\right) = -\frac{1}{2}F'\left(x_n\right)^{-1}F\left(x_n\right).$$

Then we can write

$$F'\left(x_n\right)k\left(x_n\right) = -\frac{1}{2}F\left(x_n\right) \tag{5}$$

So we can apply the Krylov method to find the approximate solution $k\left(x_n\right)$ of equation (5).

Fomula (4) is rewritten as follows

$$F'\left(x_n + k\left(x_n\right)\right)s_n = -F\left(x_n\right), \tag{6}$$

with

$$x_{n+1} = s_n + x_n. \tag{7}$$

We continue applying the Newton-Krylov algorithm to find $x_{n+1}$ of system (6), (7).

The convergence and convergence speed of the Third-order Newton-Krylov method are presented in [12], [13].

In this paper, we present a new improvement of the third order Newton-Krylov method with quaternary convergence speed and prove the convergence of the iterative formula. The structure of the paper consists of four parts as follows: The first part of the paper is Introduction. In the next section, we present the algorithm to improve the third order Newton-Krylov algorithm with quaternary convergence speed and proves the convergence. of the iterative formula. The third section presents the experimental results and Section 4 is the Conclusion.

**2. Improved algorithm**

We consider the iterative fomulation

$$x_{n+1} = x_n - \left[\frac{1}{6}F'\left(x_n\right) + \frac{2}{3}F'\left(\frac{x_n + g\left(x_n\right)}{2}\right) + \frac{1}{6}F'\left(g\left(x_n\right)\right)\right]^{-1} F\left(x_n\right), \tag{8}$$

where $g\left(x_n\right) = x_n - F'\left(x_n\right)^{-1}\left[F\left(x_n\right) + F\left(x_{n+1}^*\right)\right]$ và $x_n^* = x_n - F'\left(x_n\right)^{-1} F\left(x_n\right)$.

Then the fomulation (8) is rewritten by the following three iteration formulas

$$F'\left(x_n\right)s_n^* = -F_n\left(x_n\right) \text{ với } x_n^* = x_n + s_n^*,$$

$$F'\left(x_n\right)g_n^* = -\left[F_n\left(x_n\right) + F\left(x_{n+1}^*\right)\right] \text{ với } g\left(x_n\right) = x_n + g_n^*,$$

$$\left[\frac{1}{6}F'(x_n) + \frac{2}{3}F'\left(\frac{x_n + g(x_n)}{2}\right) + \frac{1}{6}F'\left(g(x_n)\right)\right]s_n = -F\left(x_n\right) \text{ với } x_{n+1} = x_n + s_n.$$

We use the Newton-Krylov algorithm to solve the three above systems of equation finding the solutions $s_n^*$, $g_n^*$ và $s_n$.

Next, we prove the convergence of this algorithm.

**Theorem 2. 1 ( The convergency of improved iterative fomula)** Let $F:\mathbb{R}^n \to \mathbb{R}^n$ is continuous differentiable function on a convex open set $D \subset \mathbb{R}^n$. Assume there exists $x^* \in \mathbb{R}^n$ and $\alpha, \beta > 0$ which satisfies $S\left(x^*, r\right) \subset D, F\left(x^*\right) = 0, F'(x^*)^{-1}$ exists, $\left\|F'\left(x^*\right)^{-1}\right\| \le \beta,$ and $F' \in Lip_\gamma\left(S\left(x^*, r\right)\right)$. There exists $\varepsilon > 0$ so that with each $x^0 \in S\left(x^*, \frac{\varepsilon}{2}\right)$ dãy $x_1, x_2, ..., x_n, ...$ determined by formula (8) which converges to $x^*$.

To prove Theorem 2.1, we first present the following propositions

***Lema 2.2 (Xem[14])*** *Let* $E, I \in \mathbb{R}^n$ *, where* $I$ *is the unit matrix. If* $\left\|E\right\| < 1$ *then* $\left(I - E\right)^{-1}$ *exists* *and* $\left\|(I-E)^{-1}\right\| \le \dfrac{1}{1-\|E\|}$. *Morever,if* $A$ *is a invertible matrix and* $\left\|A^{-1}(B-A)\right\| < 1$ *then* $B$ *is also a invertible matrix and* $\left\|B^{-1}\right\| \le \dfrac{\left\|A^{-1}\right\|}{1-\left\|A^{-1}(B-A)\right\|}$.

***Lema 2.3 (See[14])*** *Let the function* $F:\mathbb{R}^n \to \mathbb{R}^m$ *be a continuously differentiable map on an open convex set* $D$ *and* $F' \in Lip_\gamma(D)$, *then for every* $x+p \in D$ *we have* $\left\|F(x+p)-F(x)-F'(x)p\right\| \le \dfrac{\gamma}{2}\|p\|^2$.

Then, we will provide proof for Theorm 2.1. Given $\varepsilon = \min\left\{r, \dfrac{1}{2\beta\gamma}\right\}$.

With $\left\|x^0-x^*\right\| < \dfrac{\varepsilon}{2}, F'$ Lípchitz at $x^*$, according to Lema 2.3, we have

$$\left\|F'(x^*)^{-1}\left[F'(x^0)-F'(x^*)\right]\right\| \le \left\|F'(x^*)^{-1}\right\|\left\|F'(x^0)-F'(x^*)\right\| \le \beta\gamma\left\|x^0-x^*\right\| < \beta\gamma\frac{\varepsilon}{2} \le \frac{1}{4}.$$

Acoording to proposition 2.2 we have invertible $F'(x^0)$ and

$$\left\|F'(x^0)^{-1}\right\| \le \frac{\left\|F'(x^*)^{-1}\right\|}{1-\left\|F'(x^*)^{-1}\left[F'(x^0)-F'(x^*)\right]\right\|} < \frac{4}{3}\left\|F'(x^*)^{-1}\right\| < \frac{4}{3}\beta.$$

From the definition $x_{n+1}^*$, we have

$$x_1^*-x^* = x^0-x^*-F'(x^0)^{-1}F(x^0) = F'(x^0)^{-1}\left[F(x^*)-F(x^0)-F'(x^0)(x^*-x^0)\right].$$

Therefore

$$\left\|x_1^*-x^*\right\| \le \left\|F'(x^0)^{-1}\right\|\left\|F(x^*)-F(x^0)-F'(x^0)(x^*-x^0)\right\| \le \frac{4\beta}{3}\frac{\gamma}{2}\left\|x^0-x^*\right\|^2 < \frac{2}{3}\beta\gamma\frac{\varepsilon^2}{4} < \frac{\varepsilon}{2}.$$

Therefore we have $x_1^* \in S\left(x^*, \dfrac{\varepsilon}{2}\right)$.

From the definition $g_n(x)$, ta có $g(x^0) = x^0 - F'(x^0)^{-1}\left[F(x^0)+F(x_1^*)\right]$,therefore

$$g(x^0)-x^* = x^0-x^*-F'(x^0)^{-1}\left[F(x^0)+F(x_1^*)\right] = x_1^*-x^*-F'(x^0)^{-1}F(x_1^*)$$
$$= -F'(x^0)^{-1}\left[F(x_1^*)-F(x^0)-F'(x^0)(x^*-x^0)+F(x^*)-F(x^0)-F'(x^0)(x^*-x^0)\right].$$

Applying Lema 2.3 we have

$$\left\|g(x^0)-x^*\right\| \le \left\|F'(x^0)^{-1}\right\|\left\|F(x_1^*)-F(x^0)-F'(x^0)(x^*-x^0)\right\| + \left\|F'(x^0)^{-1}\right\|\left\|F(x^*)-F(x^0)-F'(x^0)(x^*-x^0)\right\|$$
$$\le \frac{4\beta}{3}\frac{\gamma}{2}\left[\left\|x_1^*-x^0\right\|^2+\left\|x^0-x^*\right\|^2\right] \le \frac{2}{3}\beta\gamma\left[\left\|x_1^*-x^*\right\|^2+2\left\|x_1^*-x^*\right\|\left\|x^0-x^*\right\|+\left\|x^0-x^*\right\|^2\right]$$
$$< \frac{85}{216}\beta\gamma\varepsilon^2 < \frac{\varepsilon}{2}.$$

Therefore we have $g\left(x^0\right) \in S\left(x^*, \frac{\varepsilon}{2}\right)$. ‘

Then we have $\frac{x^0 + g\left(x^0\right)}{2} - x^* = \frac{1}{2}\left(x^0 - x^* - F'\left(x^0\right)^{-1}\left[F\left(x^0\right) + F\left(x_1^*\right)\right]\right) + \frac{1}{2}\left(x^0 - x^*\right)$, therefore

$$\left\|\frac{x^0 + g\left(x^0\right)}{2} - x^*\right\| \le \frac{1}{2}\left\|x^0 - x^* - F'\left(x^0\right)^{-1}\left[F\left(x^0\right) + F\left(x_1^*\right)\right]\right\| + \frac{1}{2}\left\|x^0 - x^*\right\|$$
$$\le \frac{1}{2}\left\|g\left(x^0\right) - x^*\right\| + \frac{1}{2}\left\|x^0 - x^*\right\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Thus we conclude that $\frac{x^0 + g\left(x^0\right)}{2} \in S\left(x^*, \frac{\varepsilon}{2}\right)$.

Given $H(x) = \frac{1}{6}F'(x) + \frac{2}{3}F'\left(\frac{x + g(x)}{2}\right) + \frac{1}{6}F'\left(g(x)\right)$, then $x_{n+1} = x_n - H\left(x_n\right)^{-1} F\left(x_n\right)$.

From definition $H(x)$ we have

$$H\left(x^0\right) - F'\left(x^*\right) = \frac{1}{6}\left[F'\left(x^0\right) - F'\left(x^*\right)\right] + \frac{2}{3}\left[F'\left(\frac{x^0 + g\left(x^0\right)}{2}\right) - F'\left(x^*\right)\right] + \frac{1}{6}\left[F'\left(g\left(x^0\right)\right) - F'\left(x^*\right)\right]$$

From $\left\|x^0 - x^*\right\| < \frac{\varepsilon}{2}, F'$ Lípchitz at $x^*$ , so according to Lema 2.3 we have

$$\left\|F'\left(x^*\right)^{-1}\left[H\left(x^0\right) - F'\left(x^*\right)\right]\right\| \le$$
$$\left\|F'\left(x^*\right)^{-1}\right\| . \left[\frac{1}{6}\left\|F'\left(x^0\right) - F'\left(x^*\right)\right\| + \frac{2}{3}\left\|F'\left(\frac{x^0 + g\left(x^0\right)}{2}\right) - F'\left(x^*\right)\right\| + \frac{1}{6}\left\|F'\left(g\left(x^0\right)\right) - F'\left(x^*\right)\right\|\right]$$
$$\le \beta\left[\frac{\gamma}{6}\left\|x^0 - x^*\right\| + \frac{2\gamma}{3}\left\|\frac{x^0 + g\left(x^0\right)}{2} - x^*\right\| + \frac{\gamma}{6}\left\|g\left(x^0\right) - x^*\right\|\right] \le \beta\left(\frac{\gamma\varepsilon}{12} + \frac{\gamma\varepsilon}{3} + \frac{\gamma\varepsilon}{12}\right) \le \frac{1}{4}.$$

We have $\left\|H\left(x^0\right)^{-1}\right\| \le \frac{\left\|F'\left(x^*\right)^{-1}\right\|}{1 - \left\|F'\left(x^*\right)^{-1}\left[H\left(x^0\right) - F'\left(x^*\right)\right]\right\|} < \frac{4}{3}\left\|F'\left(x^*\right)^{-1}\right\| \le \frac{4}{3}\beta.$

On the other hand, we then have $x_1 = x_0 - H\left(x^0\right)^{-1} F\left(x^0\right)$

It follows that $x_1 - x^* = x^0 - x^* - H\left(x^0\right)^{-1} F\left(x^0\right) = H\left(x^0\right)^{-1}\left[F\left(x^*\right) - F\left(x^0\right) - F'\left(x^0\right)\left(x^* - x^0\right)\right]$.

Or

$$
\begin{aligned}
x_1 - x^* = & \frac{1}{6} H\left(x^0\right)^{-1}\left[F\left(x^*\right) - F\left(x^0\right) - F'\left(x^0\right)\left(x^* - x^0\right)\right] \\
& - \frac{2}{3} H\left(x^0\right)^{-1}\left[F'\left(\frac{x^0 + g\left(x^0\right)}{2}\right) - F'\left(x^*\right)\right]\left(x^* - x^0\right) \\
& - \frac{1}{6} H\left(x^0\right)^{-1}\left[F'\left(g\left(x^0\right)\right) - F'\left(x^*\right)\right]\left(x^* - x^0\right) \\
& - \frac{5}{6} H\left(x^0\right)^{-1}\left[F\left(x^0\right) - F\left(x^*\right) - F'\left(x^*\right)\left(x^* - x^0\right)\right].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\left\|x_1 - x^*\right\| \le & \frac{1}{6}\left\|H\left(x^0\right)^{-1}\right\|\left\|F\left(x^*\right) - F\left(x^0\right) - F'\left(x^0\right)\left(x^* - x^0\right)\right\| \\
& + \frac{2}{3}\left\|H\left(x^0\right)^{-1}\right\|\left\|F'\left(\frac{x^0 + g\left(x^0\right)}{2}\right) - F'\left(x^*\right)\right\|\left\|x^* - x^0\right\| \\
& + \frac{1}{6}\left\|H\left(x^0\right)^{-1}\right\|\left\|F'\left(g\left(x^0\right)\right) - F'\left(x^*\right)\right\|\left\|x^* - x^0\right\| \\
& + \frac{5}{6}\left\|H\left(x^0\right)^{-1}\right\|\left\|F\left(x^0\right) - F\left(x^*\right) - F'\left(x^*\right)\right\|\left\|x^* - x^0\right\|
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\|x_1 - x^*\right\| \le & \frac{\beta\gamma\varepsilon}{18}\left\|x^* - x^0\right\| + \frac{4\beta\gamma\varepsilon}{9}\left\|x^* - x^0\right\| + \frac{\beta\gamma\varepsilon}{18}\left\|x^* - x^0\right\| + \frac{5\beta\gamma\varepsilon}{18}\left\|x^* - x^0\right\| \\
& < \frac{4}{9}\left\|x^* - x^0\right\| < \frac{1}{2}\left\|x^* - x^0\right\| < \frac{\varepsilon}{2}.
\end{aligned}
$$

Thus we can conclude that $x_1 \in S\left(x^*, \frac{\varepsilon}{2}\right)$.

With similar argument, we have $x_2 \in S\left(x^*, \frac{\varepsilon}{2}\right)$ and by induction we can prove that $x_{k+1} \in S\left(x^*, \frac{\varepsilon}{2}\right)$, therefore $\left\|x_{k+1} - x^*\right\| < \left(\frac{1}{2}\right)^{k+1}\left\|x^* - x^0\right\|$, it follows that $x_n$ converges to $x^*$.

## 3. Experimential results

In this section, we give an example and by using Matlab to find the approximate solution of the system through iterative formula (8). In this example, the iterations will stop when $\left\|F\left(x_n\right)\right\| < 10^{-13}$ and we also give the running time of the algorithm.

**Example** : approximately solve the following system of equations

$$
\begin{cases}
e^{x_1} - x_3^2 - x_4^2 = 0 \\
e^{x_2} - x_4^2 - x_1^2 = 0 \\
e^{x_3} - x_1^2 - x_2^2 = 0 \\
e^{x_4} - x_2^2 - x_3^2 = 0
\end{cases}
\tag{9}
$$

By choosing an initial approximate solution $x^0 = \left(1,1,1,1\right)$, after two interations with a runtime of 0.187 (s) we have the following approximate solution of the system of equations (9)

$$\left(1.48796206549818,1.48796206549818,1.48796206549818,1.48796206549818\right).$$

**Code :**

```
clear all
syms x1 x2 x3 x4
format long;
f = [exp(x1)-x3*x3-x4*x4; exp(x2)-x4*x4-x1*x1; exp(x3)-x1*x1-x2*x2; exp(x4)-x2*x2-x3*x3];
y = [x1; x2; x3; x4]; xn = [1; 1;1; 1];
R = jacobian(f; y);
m = 0; tic;
while (m < 100)
a = subs(R,{x1; x2; x3; x4}; {xn(1); xn(2), xn(3), xn(4)});
b = -subs(f, {x1, x2, x3, x4}, {xn(1); xn(2) ), xn(3), xn(4)});
A = a' ∗ a; B = a' ∗ b;
tol = 1e - 13; z0 = zeros(2; 1);
sn∗ = fom(A; B; z0; tol);
% (Approximate solution s* for the system F'(x_n) s*_n = -F(x ) )
xn1 = n + s*_n;  % ( Calculate x*_{n+1} )
b = -(subs(f, x1, x2, x3, x4, n(1); n(2), n(3), n(4)) + subs(f, x1, x2, x3, x4, xn1(2); xn1(3), xn1(4));
A = a' ∗ a; B = a' ∗ b;
tol = 1e - 13; z0 = zeros(2; 1);
gn∗ = fom(A; B; z0; tol);  % (Compute the solution g*_n for the syste  F'(x_n)g*_n = -[F(x_n)+F(x*_{n+1})])
gn = xn + gn∗;
yn = 1/2 ∗ (n + gn);
C = 1/6 ∗ subs(R; {x1; x2; x3;x4}, {n(1); n(2); n(3); n(4)});
D = 2/3 ∗ subs(R; {x1; x2; x3;x4}; {yn(1); yn(2); yn(3); yn(4)});
E = 1/6 ∗ subs(R; {x1; x2; x3;x4}; {gn(1); gn(2); gn(3); gn(4)});
a = C + D + E;
A = a' ∗ a; B = a' ∗ b;
tol = 1e - 13; z0 = zeros(2; 1);
sn = fom(A; B; z0; tol);
% (Approximate solution s_n for the system As_n = B )
xn = xn + sn;
if norm(B) < 10-13 break;
else
```

```
m = m + 1;
end
end; toc;
fprintf('Execution time::'); disp(toc)
if (m == 100)
fprintf(' No convergence after 100 iterations');
else
fprintf('The number of iterations is:'); m
fprintf ('solution:');xn
end
```

## 4. Conclusions

The paper presents a new improvement of the third-order Newton-Krylov method with quaternary convergence speed. This is an important result so that we can solve real problems related to finding approximate solutions of complex nonlinear equations.

**Declaration of Competing Interest**

The authors declare no competing interests.

**Author contributions**

"Conceptualization: author 1 and author 2; methodology: author 1 and author 2; software: author 1; validation: author 1, author 2; formal analysis: author 2; writing original draft: author 1; review and editing: author 2; funding acquisition: author 2. All authors have read and agreed to the published version of the manuscript."

## References

[1] M. Drexler, "Newton method as a global solver for non-linear problems", *Ph.D. Thesis, University of Oxford*, 1997.

[2] S. Amat, S. Busquier, J.M. Gutierrez, "Geometric constructions of iterative methods to solve nonlinear equations", *Comp. Appl. Math*., 157 (2003) 197-205.

[3] J.M. Gutierrez, M.A. Hernandez, "A family of Chebyshev-Halley type methods in Banach spaces", *Bull. Austral. Math. Soc*., 55 (1997) 113-130.

[4] D.K.R. Babajee, M.Z. Dauhoo, "M.T. Darvishi, A. Barati, A note on the local convergence of iterative methods based on Adomian decomposition method and 3-node quadrature rule", *Appl. Math. Comput.*, 200 (2008) 452–458.

[5] D.K.R. Babajee, M.Z. Dauhoo, "An analysis of the properties of the variants of Newton's method with third order convergence", *Appl. Math. Comput*., 183 (2006) 659-684.

[6] D.K.R. Babajee, M.Z. Dauhoo, "Analysis of a family of two-point iterative methods with third order convergence", *International Conference on Numerical Analysis and Applied Mathematics* (2006), pp. 658-661.

[7] A. Cordero, J.R. Torregrosa, "Variants of Newton's method for functions of seversl variables", *Appl. Math. Comput*., 183 (2006) 199-208.

[8] M.T. Darvishi, A. Barati, "A third-order Newton-type method to solve systems of nonlinear equations", *Appl.Math. Comput*., 187 (2007) 630-635.

[9] M.T. Darvishi, A. Barati, "Super cubic iterative methods to solve systems of nonlinear equations", *Appl. Math. Comput*., 188 (2007) 1678-1685.

[10] Ioannis K. Argyros, "Convergence and Applications of Newton type iterations", *Springer Science + Business Media LLC, 233 Spring Street New York*, NY 10013, U.S.A, 2008.

[11] M.T. Darvishi and Byeong – Chun Shin, "High –Order Newton – Krylov Methods to Solve systems of Nonlinear Equation", *J.KIAM, (Vol. 15, No.1),* 19 -30, 2011.

[12] Lai Van Trung, Hoang Phuong Khanh, Quach Thi Mai Lien, Nguyen Viet Hoang, “Solving system of nonlinear equations by the third – oder Newton-Krylov method”, *TNU Journal of Science and Technology*, 225 (06), 405-410, 2020.

[13] Lai Van Trung, Quach Thi Mai Lien, “The speed of convergence of the third-oder Newton-Krylov method”, *TNU Journal of Science and Technology*, 226 (07), 50-55, 2021.

[14]. J. E. Dennis, Jr. and Robert B. Schnabel, “Numerical Methods for Unconstrained Optimization and Nonlinears”, *Prentice Hall, Inc., Englewood Cliffs*, 1983.