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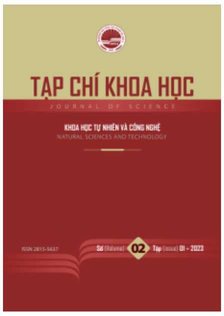

Article type: Research article

On the puiseux theorem

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Abstract

In 1850, Puiseux solved the problem of finding roots of complex polynomials in two variables and proved that the field of these series is algebraically closed. His proof provided an algorithm constructing the roots.

In this article, based on the paper “Ha Huy Vui, Nguyen Hong Duc. *On the Lojasiewicz exponent near the fibre of polynomial mappings*, Ann. Polon. Math. 94 (2008), 43-52”, we give a different algorithm computing Newton - Puiseux roots of a complex polynomial in two variables. This algorithm is more effective in practice.

Keywords: “the Puiseux Theorem”, “the Puiseux theorem”.

1. Introduction

As a continuation of the classical problem of finding all roots of a complex polynomial, Puiseux Theorem gives an algorithm looking for roots of polynomial in two variables. It quickly became a powerful tool in many areas of mathematics such as algebra, semi-algebraic, number theory. Let $f(x,y) \in \mathbb{C}[x,y]$ be a complex polynomial. We may consider $f(x,y) \in \mathbb{C}[x][y]$ as a polynomial of one variable y with coefficients in the ring $\mathbb{C}[x]$. A classical problem in mathematics is to find roots of f. In [1], [3], [4], Puiseux gave an algorithm finding all roots $y = y_i\left(x\right)$ of $f\left(x,y\right) = 0$. In this article based on [2], we give a different algorithm computing the roots $y_i\left(x\right)$. This method is easier in practice as Example 4.5, 4.6, 4.7 illustrated. The article is organized as follows.

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1. We will present overview about the Puiseux series.
2. The next part of the article, we give an inductive algorithm, named the Newton - Puiseux algorithm, which gives all y-roots of f. Recall the definition of the Newton polygon of f and give some examples to represent them on diagram.
3. In last section, we will give the main result of the paper which is Puiseux's theorem.

2. Puiseux series

Let $\mathbb{C}[[x_1,...,x_n]]$ be ring of formal power series.

Definition 2.1. A *Puiseux series* over $\mathbb{C}$ is a series of the form:

$$a=\varphi(x)=c_1x^{n_1/N}+c_2x^{n_2/N}+...,$$

where $c_1\neq 0, c_i\in\mathbb{C}, n_i, N\in\mathbb{Z}, n_1<n_2<..., N>0.$

Let n_1/N be called *the order* of φ and denoted by $\operatorname{ord}(\varphi)$.

The *ring of Puiseux series* over $\mathbb{C}$ is denoted by $\mathbb{C}\langle\langle x\rangle\rangle$.

Assume that

$$\varphi(x)=c_1x^{n_1/N}+c_2x^{n_2/N}+...=\sum_{i\geq 1}^{\infty}c_ix^{n_i/N},$$

and

$$\psi(x)=b_1x^{m_1/M}+b_2x^{m_2/M}+...=\sum_{i\geq 1}^{\infty}b_ix^{m_i/M},$$

are any two Puiseux series. The addition, multiplication operations in $\mathbb{C}\langle\langle x\rangle\rangle$ are defined as the polynomial rings.

Remark 2.2. We see that $\varphi(y^N)=c_1y^{n_1}+c_2y^{n_2}+...\in\mathbb{C}[[x]]$.

Lemma 2.3. $(\mathbb{C}\langle\langle x\rangle\rangle,+,\cdot)$ is a field.

Proof. We only need to prove that for all $\varphi(0)\neq 0$ then $\varphi(x)$ is invertible, i.e, there exists $\psi(x)\in\mathbb{C}\langle\langle x\rangle\rangle$ such that $\varphi(x).\psi(x)=1$.

Assume that $\varphi(x)=c_1x^{n_1/N}+c_2x^{n_2/N}+...,$ where $c_i\in\mathbb{C}, c_1\neq 0, n_i, N\in\mathbb{Z}, n_1<n_2<..., N>0.$

Let $y=x^{1/N}$ where N is the common denominator of all exponents in the Puiseux series. Then

$$\begin{aligned}\varphi(x)=\varphi(y^N)&=c_1y^{n_1}+c_2y^{n_2}+...\in\mathbb{C}[[x]]\\&=y^{n_1}\left(c_1+c_2y^{n_2-n_1}+...\right)\\&=y^{n_1}\tilde{\varphi}(y).\end{aligned}$$

We have $\tilde{\varphi}(0)=c_1\neq 0$. Then there exist $\tilde{\psi}(y)\in\mathbb{C}[[x]]$ such that (proved in Lemma 2.4). Define

$$\psi(x)=x^{-n_1/N}\tilde{\psi}(x^{1/N}).$$

Then,

$$\varphi(x).\psi(x)=\varphi(y^N).\psi(y^N)=y^{n_1}\tilde{\varphi}(y).x^{-n_1/N}\tilde{\psi}(x^{1/N})=1.$$

Lemma 2.4. Let $\tilde{\varphi}(x)\in\mathbb{C}[[x]]$ such that $\tilde{\varphi}(0)\neq 0$. Then $\tilde{\varphi}$ is invertible.

Proof. Let $\mathbb{C}[[x]]$ be the ring of formal power series and let us write $\tilde{\varphi}(x)$ as $a_0+a_1x+..., a_0\neq 0$ and $\tilde{\psi}(x)$ as $b_0+b_1x+...$. We will inductively construct a series $\tilde{\psi}(x)$ satisfying $\tilde{\varphi}(x)\tilde{\psi}(x)=1$. We have:

$$\begin{aligned}\tilde{\varphi}(x)\tilde{\psi}(x)&=a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2+...\\&=c_0+c_1x+...+c_nx^n+...=1.\end{aligned}$$

Then $b_0=\dfrac{1}{a_0}$ and $b_1=-\dfrac{a_1b_0}{a_0}=-\dfrac{a_1}{a_0^2},...$

Assume that we have obtained $b_0,...,b_n$ such that

$$c_0=1, c_1=...=c_n=0.$$

We defined:

$$b_{n+1}=-\frac{a_1b_n+...+a_{n+1}b_0}{a_0}.$$

Then

$$a_0b_{n+1}+a_1b_n+...+a_{n+1}b_0=0.$$

Hence, there exists $\tilde{\psi}(x)$ such that $\tilde{\varphi}(x)\tilde{\psi}(x)=1$.

3. Newton polygon

In the section, we will present an inductive algorithm, named the Newton - Puiseux algorithm, which gives all y-roots of f.

Consider $f(x,y)\in\mathbb{C}\langle\langle x\rangle\rangle[[y]]$ and $f(x,y)=\sum c_n(x)y^n=\sum_{(\alpha,\beta)}c_{\alpha\beta}x^\alpha y^\beta$.

Definition 3.1. Let $\text{Supp}(f):=\left\{(\alpha,\beta)\in\mathbb{Q}\times\mathbb{N}\,\middle|\,c_{\alpha\beta}\neq 0\right\}$ be the support of f.

Let $\Delta_-(f)$ be convex hull of the set

$$\left\{(\alpha,\beta)+\mathbb{R}_+^2\,\middle|\,(\alpha,\beta)\in\text{Supp}(f)\right\}.$$

Let Γ_f be the union of compact edges of $\Delta_-(f)$, we call it the Newton polygon (diagram) of f .

Example 3.2. Let $f(x,y)=3x^2y^4-x^{-1/2}y^3+7x^{2/3}y^5+x^3+y$.

We have:

$$\Delta_-(f)=\left\{(2;4),\left(-\frac{1}{2};3\right),\left(\frac{2}{3};5\right),(3;0),(0;1)\right\}.$$

Γ_f is the polylines connecting the points $\left(-\frac{1}{2};3\right),(0;1)$ and $(3;0)$.

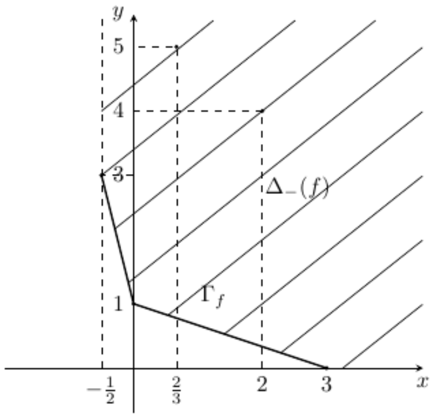

Figure 3.1. The Newton polygon of f.

Example 3.3. Let $f(x,y) = xy^3 - 2x^{-1/2}y^5 + 5x^{3/5}y + x^2$.

We have:

$$\Delta_-(f) = \left\{(1;3),\left(\frac{-1}{2};5\right),\left(\frac{3}{5};1\right),(2;0)\right\}.$$

Γ_f is the polylines connecting the points $\left(\frac{-1}{2};5\right),\left(\frac{3}{5};1\right)$ and $(2;0)$.

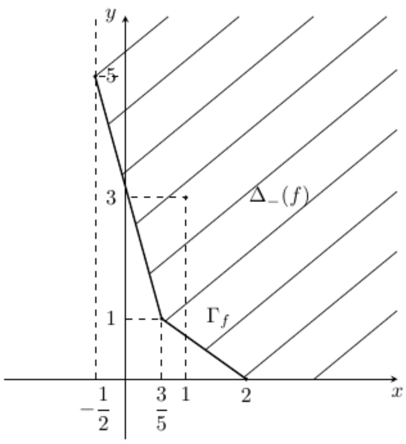

Figure 3.2. The Newton polygon of f.

4. Main result

In this section, we state the Puiseux Theorem saying that the field of Newton - Puiseux series is algebraically closed. Equivalently, one can find the roots of the equation $f(x,y)=0$ in the form $y = y(x)$ where $y(x)$ are Puiseux series. This has proven in [3], [4]. The proof provide an algorithm to find the roots $y = y(x)$ of $f(x,y)=0$. We introduce a new algorithm also computing the roots of the equation. It based on the "sliding method" introduced in [5] and developed in [2].

Theorem 4.1. *(Puiseux)* Let $f \in \mathbb{C}[[x,y]]$ such that $\operatorname{ord} f(0,y) = m$ then

$$f = u(x,y).\prod_{i=1}^{m}\left(y - y_i(x)\right),$$

where $u\left(x,y\right)$ is invertible in $\mathbb{C}[[x,y]]$ and $y_i(x)$ is Puiseux series in $\mathbb{C}\langle\langle x\rangle\rangle$. The series $y_i(x)$ are called *Newton - Puiseux roots of* f.

Corollary 4.2. The field $\mathbb{C}\langle\langle x\rangle\rangle$ is algebraically closed.

We will not prove the theorem but we will give an algorithm for constructing the solution $s(x)$. The algorithm is based on the paper [2], [5]. The algorithm as follow:

$$f(x) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} y^{\beta}.$$

- *The first step:* Construct $\mathrm{P}_{\mathrm{E}_1}(y)$:

+ Let Γ_f be the Newton polygon of f and let E_1 be the edge of Γ_f containing the lowest dot on $\mathrm{O}x$ of Γ_f.

+ Let $\mathrm{P}_{\mathrm{E}_1}(x,y) = \sum_{(\alpha,\beta)\in \mathrm{E}_1} c_{\alpha,\beta} x^{\alpha} y^{\beta}$ and $\mathrm{P}_1(y) = \mathrm{P}_{\mathrm{E}_1}(1,y) = 0$.

- *The second step:* Find a_1 and θ_1 with the following properties:

Let a_1 be a root of $\mathrm{P}_1(y)$ and $\theta_1 = \tan\gamma_1 > 0$ where θ_1 is tangent value of the angle γ_1 determining by Oy and E_1.

- *The third step:* Determine $f_1(x,y) = f(x, y + a_1 x^{\theta_1})$.

- *The four step:*

+ Applying the third step, we get a_2, θ_2 and f_2.

+ Repeating this work (infinity) many times we obtain a sequence

$$s_n(x) = a_1 x^{\theta_1} + a_2 x^{\theta_2} + \ldots + a_n x^{\theta_n}$$

with

$$\begin{aligned} f_n(x,y) &= f_{n-1}\left(x, y + a_n x^{\theta_n}\right) \\ &= f\left(x, y + s_n(x)\right). \end{aligned}$$

- *The final step:* We define

$$s(x) = \lim s_n(x).$$

The following lemma gives us an observation that $s(x)$ would be a root of f. The fact that $s(x)$ is indeed a Newton - Puiseux root of f can be found in [1].

Lemma 4.3. One has $\operatorname{ord} f(x,0) < \operatorname{ord} f(x, s_1(x)) < \ldots < \operatorname{ord} f(x, s_n(x))$.

Proof. We need only prove that $\operatorname{ord} f(x,0) < \operatorname{ord} f(x, s_1(x))$, i.e., $\operatorname{ord} f(x,0) < \operatorname{ord} f(x, a_1 x^{\theta_1})$, since the other inequalities are proved similarly. Because

$$f(x,y)=\sum_{\alpha,\beta} c_{\alpha\beta}x^{\alpha}y^{\beta},$$

then $f(x,0)=\sum_{\alpha} c_{\alpha,0}x^{\alpha}$, so $\text{ord}f(x,0)=\text{ord}\left(\sum_{\alpha} c_{\alpha,0}x^{\alpha}\right)=\alpha_{E_1}$ where $(\alpha_{E_1},0)$ is a vertex of E_1,

according to definition of E_1. We have:

$$\begin{aligned} f_1(x,y) &= f(x,y+a_1x^{\theta_1}) \\ &= \sum_{\alpha,\beta} c_{\alpha\beta}x^{\alpha}\left(y+a_1x^{\theta_1}\right)^{\beta} \\ &= \sum_{\alpha,\beta} c_{\alpha\beta}x^{\alpha}\sum_{0\le k\le\beta} C_{\beta}^{k}y^{\beta-k}\left(a_1x^{\theta_1}\right)^{k} \\ &= \sum_{\alpha,\beta} c_{\alpha\beta}x^{\alpha}\sum_{0\le k\le\beta} C_{\beta}^{k}a_1^{k}y^{\beta-k}x^{k\theta_1}. \end{aligned}$$

We see that the term $C_{\beta}^{k}a_1^{k}y^{\beta-k}x^{k\theta_1}$ vanishes if $y=0$ and $\beta<k$, therefore

$$\begin{aligned} f_1(x,0) &= \sum_{\alpha,\beta} c_{\alpha\beta}x^{\alpha}a_1^{\beta}x^{\beta\theta_1} \\ &= \sum_{(\alpha,\beta)\in \text{Supp}(f)} c_{\alpha\beta}a_1^{\beta}x^{\alpha+\beta\theta_1}. \end{aligned}$$

Hence, $\text{ord}f_1(x,0)\ge\alpha_{E_1}$ due to Lemma 4.4. The coefficient of $x^{\alpha_{E_1}}y^{0}$ of $f_1(x,0)$ is

$$\sum_{\alpha+\beta\theta_1=\alpha_{E_1}} c_{\alpha\beta}a_1^{\beta}=P_{E_1}(a_1)=0.$$

Then, $\text{ord}f_1(x,0)>\alpha_{E_1}$, i.e, $\text{ord}f_1(x,0)>\text{ord}f(x,0)$. Therefore,

$$\text{ord}f(x,0)<\text{ord}f(x,s_1(x))<\ldots<\text{ord}f(x,s_n(x)).$$

Lemma 4.4. Let $(\alpha,\beta)\in\text{Supp}(f)$ then $\alpha+\beta\theta_1\ge\alpha_{E_1}$.

Proof.

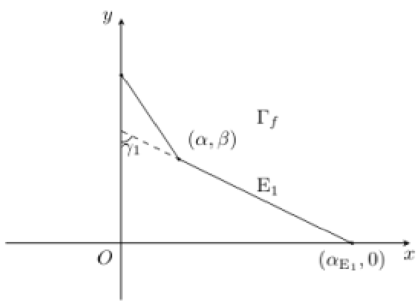

Figure 4.1. The Newton polygon of f.

Let $\theta_1=\tan\gamma_1>0$ where θ_1 is the tangent value of the angle γ_1 determining by Oy and E_1.

From the above diagram, we have $\theta_1 = \tan\gamma_1 = -k$ where k is the slope of E_1 and Oy. So, the equation of the line containing E_1 passing through a point $(\alpha_{E_1}, 0)$ with slope $k = -\theta_1$:

$$x + \theta_1 y - \alpha_{E_1} = 0.$$

We see that the origin O does not belong to the semi-plane containing the support of f and that

$$0 + 0 - \alpha_{E_1} < 0.$$

Hence, $\alpha + \beta\theta_1 \geq \alpha_{E_1}, \forall (\alpha, \beta) \in \mathrm{Supp}(f)$.

Example 4.5. Find a (truncated) Newton - Puiseux root of $f(x, y) = x^3 + x^4 + y^2$.

We have:

$$\Delta_-(f) = \{(3;0), (4;0), (0;2)\}.$$

Let E_1 be the edge of Γ_f containing the lowest dot on Ox. Then $P_{E_1}(x, y) = x^3 + y^2$ and $P_{E_1}(1, y) = 1 + y^2$. Hence, $P_{E_1}(1, y) = 0 \Leftrightarrow y = \pm i$ .Then, we have the following Newton polygon of f.

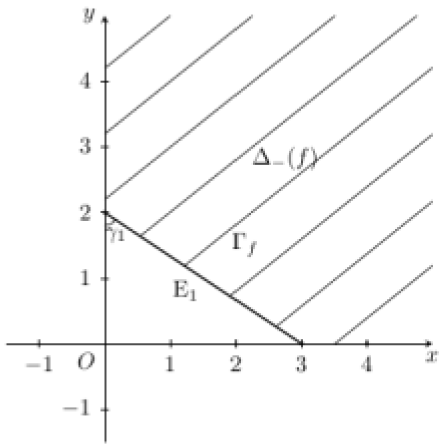

Figure 4.2. The Newton polygon of f.

Let a_1 be a root of $f(y)$ and θ_1 be the tangent value of the angle γ_1 determining by E_1 and Oy, so $a_1 = i$ and $\theta_1 = \tan\gamma_1 = \dfrac{3}{2}$. We have $s_1(x) = ix^{3/2}$ and

$$\begin{aligned} f_1(x, y) &= f\left(x, y + ix^{3/2}\right) \\ &= x^3 + x^4 + \left(y + ix^{3/2}\right)^2 \\ &= x^4 + y^2 + 2ix^{3/2}y. \end{aligned}$$

Let E_2 be the edge of Γ_{f_1} containing the lowest dot on Ox. Then $P_{E_2}(x, y) = x^4 + 2ix^{3/2}y$ and $P_{E_2}(1, y) = 1 + 2iy$. Hence,

$$\mathrm{P}_{\mathrm{E}_2}(1, y) = 0 \Leftrightarrow y = \frac{-1}{2i} = \frac{i}{2}.$$

We have the following Newton polygon of f_1.

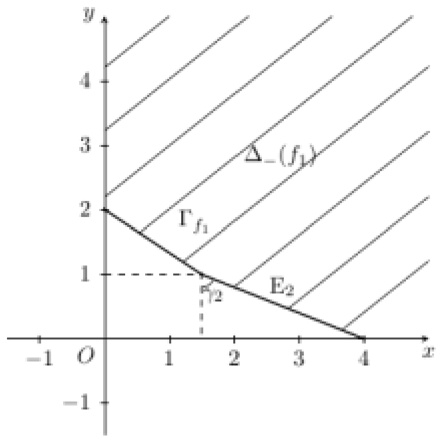

Figure 4.3. The Newton polygon of f_1.

Let a_2 be a root of $f_1(y)$ and θ_2 be the tangent value of the angle γ_2 determining by E_2 and $x = \frac{3}{2}$, so $a_2 = \frac{i}{2}$ and $\theta_2 = \tan\gamma_2 = \frac{5}{2}$. We have $s_2(x) = \frac{i}{2}x^{5/2}$ and

$$\begin{aligned} f_2(x,y) &= f_1\left(x, y + \frac{i}{2}x^{5/2}\right) \\ &= x^4 + \left(y + \frac{i}{2}x^{5/2}\right)^2 + 2ix^{3/2}\left(y + \frac{i}{2}x^{5/2}\right) \\ &= x^4 + y^2 - \frac{1}{2}x^5 + ix^{5/2}y + 2ix^{3/2}y - x^4 \\ &= y^2 - \frac{1}{2}x^5 + ix^{5/2} + 2ix^{3/2}y. \end{aligned}$$

Let E_3 be the edge of Γ_{f_2} containing the lowest dot on $\mathrm{O}x$. Then $\mathrm{P}_{\mathrm{E}_3}(x, y) = -\frac{1}{2}x^5 + 2ix^{3/2}y$ and $\mathrm{P}_3(y) = \mathrm{P}_{\mathrm{E}_3}(1, y) = \frac{1}{2} + 2iy$. Hence,

$$\mathrm{P}_3(y) = 0 \Leftrightarrow y = \frac{1}{4i}.$$

We have the following Newton polygon of f_2.

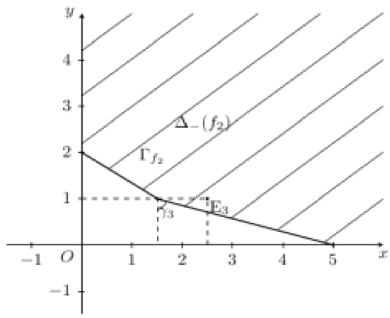

Figure 4.4. The Newton polygon of f_2.

Let a_3 be a root of $f_2(y)$ and θ_3 be the tangent value of the angle γ_3 determining by E_3 and $x = \frac{3}{2}$, so $a_3 = \frac{1}{4i}$ and $\theta_2 = \tan\gamma_2 = \frac{7}{2}$. We have $s_3(x) = \frac{1}{4i}x^{5/2}$. Therefore,

$$s(x) = ix^{3/2} + \frac{i}{2}x^{5/2} + \frac{1}{4i}x^{5/2} + ...$$

Example 4.6. Find a (truncated) Newton - Puiseux root of $f(x,y) = y^2 - 2xy + x^2 - x^2y - 5x^3$

We have:

$$\Delta_-(f) = \{(0;2),(1;1),(2;0),(2;1),(3;0)\}.$$

Let E_1 be the edge of Γ_f containing the lowest dot on Ox. We get: $P_{E_1}(x,y) = y^2 - 2xy + x^2$ and $P_1(y) = P_{E_1}(1,y) = y^2 - 2y + 1$. Hence,

$$P_1(y) = 0 \Leftrightarrow y = 1.$$

Then, we have the following Newton polygon of f.

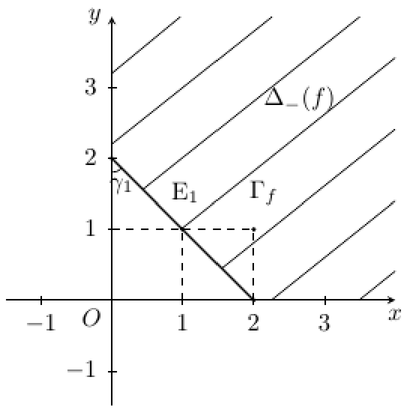

Figure 4.5. The Newton polygon of f.

Let a_1 be a root of $f(y)$ and θ_1 be the tangent value of the angle γ_2 determining by E_1 and Oy, so $a_1 = 1$ and $\theta_1 = \tan\gamma_1 = 1$. We have $s_1(x) = x$ and

$$\begin{aligned} f_1(x,y) &= f(x, y+x) \\ &= (y+x)^2 - 2x(y+x) + x^2 - x^2(y+x) - 5x^3 \\ &= y^2 - x^2 y - 6x^3. \end{aligned}$$

Let E_2 be the edge of Γ_{f_1} containing the lowest dot on $\mathrm{O}x$. Then $\mathrm{P}_{\mathrm{E}_2}(x,y) = y^2 - 6x^3$ and $\mathrm{P}_2(y) = \mathrm{P}_{\mathrm{E}_2}(1,y) = y^2 - 6$. Hence,

$$\mathrm{P}_2(y) = 0 \Leftrightarrow y = \pm\sqrt{6}\,.$$

We have the following Newton polygon of f_1.

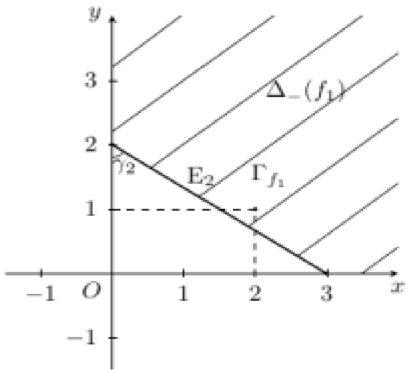

Figure 4.6. The Newton polygon of f_1.

Let a_2 be a root of $f_1(y)$ and θ_2 be the tangent value of the angle γ_2 determining by E_2 and Oy, so $\theta_2 = \tan\gamma_2 = \dfrac{3}{2}$. We have $s_2(x) = \sqrt{6}x^{3/2}$ and

$$\begin{aligned} f_2(x,y) &= f_1\left(x, y + \sqrt{6}x^{3/2}\right) \\ &= \left(y + \sqrt{6}x^{3/2}\right)^2 - x^2\left(y + \sqrt{6}x^{3/2}\right) - 6x^3 \\ &= y^2 + 2\sqrt{6}x^{3/2}y - x^2 y - \sqrt{6}x^{7/2}. \end{aligned}$$

Let E_3 be the edge of Γ_{f_2} containing the lowest dot on $\mathrm{O}x$. We have the following Newton polygon of f_2.

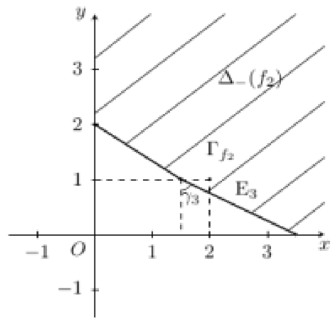

Figure 4.7. The Newton polygon of f_2.

Then $P_{E_3}(x,y)=2\sqrt{6}x^{3/2}y-\sqrt{6}x^{7/2}$ and $P_3(y)=P_{E_3}(1,y)=2\sqrt{6}y-\sqrt{6}$. Hence,

$$P_3(y)=0\Leftrightarrow y=\frac{1}{2}.$$

Let a_3 be root of $f_2(y)$ and θ_3 the tangent value of the angle γ_2 determining by E_3 and Oy, so $\theta_3=\tan\gamma_3=2$. We have $s_3(x)=\frac{1}{2}x^2$. Therefore,

$$s(x)=x+\sqrt{6}x^{3/2}+\frac{1}{2}x^2+\dots$$

Example 4.7. Find a (truncated) Newton - Puiseux root of $f(x,y)=4x^4+4x^2y+xy^2+y^3$.

We have:

$$\Delta_-(f)=\{(4;0),(2;1),(1;2),(0;3)\}.$$

Let E_1 be the edge of Γ_f containing the lowest dot on Ox. Then $P_{E_1}(x,y)=4x^4+4x^2y$ and $P_{E_1}(1,y)=4+4y$. Hence, $P_{E_1}(1,y)=0\Leftrightarrow y=-1$. Then, we have the following Newton polygon of f.

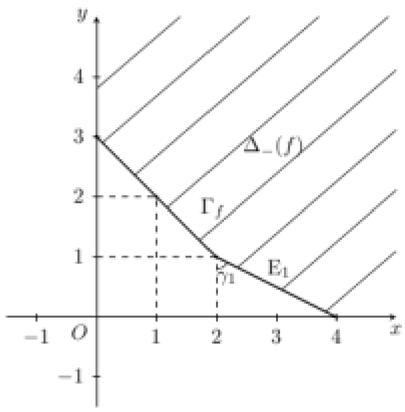

Figure 4.8. The Newton polygon of f.

Let a_1 be a root of $f(y)$ and θ_1 be the tangent value of the angle γ_2 determining by E_1 and $x=2$, so $a_1=-1$ and $\theta_1=\tan\gamma_1=2$. We have $s_1(x)=-x^2$ and

$$\begin{aligned} f_1(x,y) &= f\left(x,y-x^2\right) \\ &= 4x^4+4x^2\left(y-x^2\right)+x\left(y-x^2\right)^2+y^3 \\ &= x^5-2x^3y+4x^2y+xy^2+y^3. \end{aligned}$$

Let E_2 be the edge of Γ_{f_1} containing the lowest dot on Ox.

Then $P_{E_2}(x,y)=x^5+4x^2y$ and $P_{E_2}(1,y)=1+4y$. Hence, $P_{E_2}(1,y)=0\Leftrightarrow y=-\frac{1}{4}$. We have the following Newton polygon of f_1.

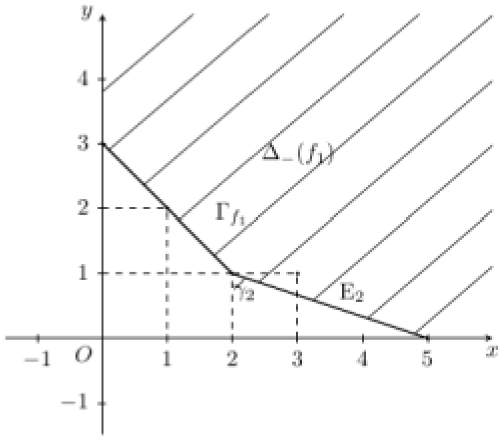

Figure 4.9. The Newton polygon of f_1.

Let a_2 be a root of $f_1(y)$ and θ_2 be the tangent value of the angle γ_2 determining by E_2 and $x=2$, so $a_2=-\frac{1}{4}$ and $\theta_2=\tan\gamma_2=3$. We have $s_2(x)=-\frac{1}{4}x^3$ and

$$f_2(x,y)=f_1\left(x,y-\frac{1}{4}x^3\right)$$
$$=\frac{1}{64}x^9+\frac{1}{16}x^7+\frac{1}{2}x^6+\frac{3}{16}x^6y-\frac{1}{2}x^4y-2x^3y-\frac{3}{4}x^3y^2+4x^2y+xy^2+y^3.$$

Let E_3 be the edge of Γ_{f_2} containing the lowest dot on Ox. Then $P_{E_3}(x,y)=4x^2y+\frac{1}{2}x^6$ and $P_{E_3}(1,y)=4y+\frac{1}{2}$. Hence,

$$P_{E_3}(1,y)=0\Leftrightarrow y=-\frac{1}{8}.$$

We have the following Newton polygon of f_2.

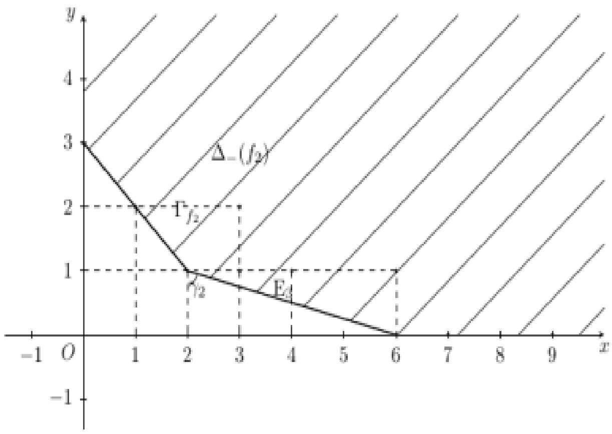

Figure 4.10. The Newton polygon of f_2.

Let a_3 be a root of $f_2(y)$ and θ_3 be the tangent value of the angle γ_2 determining by E_3 and $x = 2$, so $a_3 = -\frac{1}{8}$ and $\theta_3 = \tan\gamma_3 = 4$. We have $s_3(x) = -\frac{1}{8}x^4$. Therefore,

$$s(x) = -x^2 - \frac{1}{4}x^3 - \frac{1}{8}x^4 + ...$$

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5. Conclusions

In this article, we give a different algorithm computing Newton - Puiseux roots of a complex polynomial in two variables. This method is easier to use and more effective in practice. We also give some illustrative examples and describe the algorithm to find a (truncated) Newton - Puiseux root of these equation.

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