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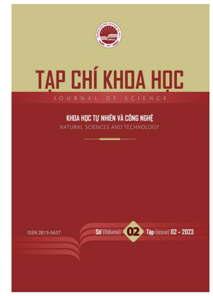

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Criteria for finite-time stability of singular large-scale discrete-time delay systems

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Abstract

This paper concerns a problem of finite-time stability for a class of linear singular large-scale systems with delays. Based on matrix transformations, Lyapunov function method combined with new estimation techniques, we derive sufficient conditions for solving the finite-time stability of the system. A numerical example is given to illustrate the validity and effectiveness of the theoretical results.

Keywords: Schur's complement lemma, finite-time stability, large-scale system, singular system.

1. Introduction

Currently, the research on the stability of dynamical systems has received attention and development as an independent mathematical theory with numerous applications in scientific, engineering, and economic fields ([2],[5],[18]). The concept of finite-time stability (FTS) is independent to Lyapunov stability and was first introduced by Russian mathematicians ([9]), appearing in Western journals in the 1960s ([3]). In comparison to Lyapunov stability- addresses the behavior of a system over an infinite time interval, finite-time stability focuses on the boundedness of a system within a fixed, generally short, time interval. Therefore, finite-time stability (FTS) is often used to indicate when the state variables of a system do not exceed a given threshold within a short time period, for example, preventing the system from reaching saturation or excitatory states in nonlinear dynamical systems...

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Although the theory of Lyapunov stability for linear systems with delays has been extensively developed over several decades, there are only a few results concerning the finite-time stability of linear systems with delays ([1],[6],[10]),… Most of the research results have focused on linear systems without delays or linear systems with delays but without singularity. In practice, studying the stability problem and control stability (stabilization) of singular systems with delay is often more complex than the nonsingular case due to algebraic constraints. Establishing the existence and uniqueness of solutions for these problems is much more challenging compared to the nonsingular class of problems. Moreover, delays themselves pose obstacles in the study of system stability. Therefore, the problem of investigating the finite-time stability of the class of singular, time-delay control systems has attracted considerable attention from mathematicians such as Myskist, Amato, Kharitonov, etc. In Vietnam, professors Nguyen Khoa Son, Vu Hoang Linh, Pham Huu Anh Ngoc, Vu Ngoc Phat,.. and their colleagues have also conducted research and obtained significant results ([4],[11],[15]).

Recently, there have been several published results on the stability analysis of large-scale systems that have attracted the attention of mathematicians. Many real-world systems are modeled as large-scale systems, such as power systems, communication systems, social systems, transportation systems, and economic systems. Large-scale systems are systems that consist of numerous interconnected subsystems in a tightly coupled and complex manner ([16]). The analysis of stability of large-scale systems, especially singular large-scale systems with delays becomes more challenging not only due to the high dimensionality of the systems but also because of the singularity and time-delay characteristics of the systems under investigation. Some stability results for this class of systems have been published, primarily focusing on Lyapunov stability (LS) ([12],[14],[17]). However, there are very few studies on finite-time stability (FTS) for this class of systems, mainly limited to nonsingular systems with constant delays or nonsingular systems with bounded time-varying delays ([13]). There are few publications on the finite-time stability (FTS) of singular large-scale systems with delays. Most recently, V. N. Phat and colleagues have published some results on the finite-time stability and finite-time stabilization of singular, time-delayed large-scale systems in both continuous-time ([7]) and discrete-time ([8]) cases.

With the development of digital computers, the theory of discrete systems plays a crucial role in control theory. In practical systems, discrete-time systems often arise as a result of sampling continuous-time systems, using available discrete data, or when computers are involved in the control loop. Discrete-time systems are prevalent in social systems, time series analysis, and many other real-world applications. Therefore, the study of finite-time stability and control problems for discrete systems is highly relevant, especially for complex systems that model various real-world systems such as large-scale systems. It is recognized that this is a topic of interest for many mathematicians worldwide, including those in Vietnam. At the same time, there are still many open issues for us to investigate. Building upon the results in [18] by Wu et al. for general discrete systems, we conducted a study to establish conditions for the finite-time stability of complex singular, time-delayed large-scale discrete systems.

The remainder of this paper is organized as follows: In Section 2, some preliminaries are given. Criteria for finite-time stability of singular large-scale discrete-time delay systems are constructed in Section 3. An illustrative example is contributed in Section 4 and finally, Section 5 is presented a conclusion.

2. Preliminaries

Consider a singular linear large-scale discrete system with delays of the form

$$\begin{cases} E_i x_i(k+1) = A_i x_i(k) + \sum_{j=1, j\neq i}^{N} A_{ij} x_j(k-\ell_{ij}), & k \in \mathbb{Z}^+, \\ x_i(k) \qquad = \psi_i(k), \quad k = -\ell, -\ell+1, \ldots 0, \end{cases} \tag{1}$$

where $\ell_{ij} > 0; \ell = \max\{\ell_{ij}\}; x_i(k) \in \mathbb{R}^{n_i}$ is the state; E_i is singular, $\operatorname{rank} E_i = r_i, i = \overline{1,N}$; $A_i \in \mathbb{R}^{n_i \times n_i}, A_{ij} \in \mathbb{R}^{n_i \times n_j}$ are given constant matrices; $\psi_i(k) \in \mathbb{R}^{n_i}$ are the initial functions.

Definition 1 ([8])

(i) Large-scale system (1) is said to be regular if $\det(sE_i - A_i)$ is not identical zero, for $i = \overline{1,N}$, for some $s \in \mathbb{C}$.

(ii) Large-scale system (1) is said to be causal if $\deg(\det(sE_i - A_i)) = r_i = \operatorname{rank} E_i; i = \overline{1,N}$, for some $s \in \mathbb{C}$.

As shown in [8], the regularity and causality of $(E_i, A_i), i = \overline{1,N}$, guarantee the existence and uniqueness of solutions of system (1) under admit initial condition $\psi_i(.) \in \mathbb{R}^{n_i}$.

Let us set

$$R = \operatorname{diag}\{R_1, \ldots, R_N\},\ x(k) = \operatorname{col}\{x_1(k), \ldots, x_N(k)\}, \psi(t) = \operatorname{col}\{\psi_1(k), \ldots, \psi_N(k)\}.$$

Definition 2 (Finite-time stability-FTS) The system (1) is finite-time stable with respect to $(c_1, c_2, T, R_1, \ldots, R_N)(0 < c_1 < c_2, T > 0, R_i > 0, i = \overline{1,N})$, if it is causal, regular and

$$\max_{k=-\ell,\ldots,0} \{\psi^\top(k) R \psi(k)\} \leq c_1 \to x^\top(k) R x(k) < c_2, \quad \forall k = 0, \ldots, T.$$

The aims of this paper are to find some sufficient conditions which guarantee the system (1) is regular, causal and state bounded over the finite interval $[0, T]$. We present the following propositions which will be used in the proof of the further results.

Proposition 1 *(Cauchy matrix inequality* [18]*) For given $a, b \in \mathbb{R}^n$, we have*

$$2a^\top b \leq a^\top R a + b^\top R^{-1} b,$$

where R is a positive symmetric defined matrix.

Proposition 2 *(Schur complement lemma* [18]*) For U, V, Q, where $V = V^\top > 0, U = U^\top$ are given matrices, we have*

$$U + Q^\top V^{-1} Q < 0 \Leftrightarrow \begin{bmatrix} U & Q^\top \\ Q & -V \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -V & Q \\ Q^\top & U \end{bmatrix} < 0.$$

3. Main result

In this section, some sufficient conditions for FTS will be established based on the LMI approach for system (1). We define the following matrix notations:

$$\begin{aligned}
\pi^i_{i,i} &= \eta(N-1)Q_i - X_i^\top (A_i - E_i) - (A_i - E_i)^\top X_i;\\
\pi^i_{i,k} &= -X_i A_{ik} + (A_i - E_i)^\top Z_i^\top A_{ik};\ k = 1,\dots,i-1,i+1,\dots,N;\\
\pi^i_{i,N+1} &= E_i^\top P_i + (A_i - E_i)^\top Y_i^\top + X_i;\\
\pi^i_{k,k} &= -\eta Q_k + A_{ik}^\top Z_i A_{ik} + A_{ik}^\top Z_i^\top A_{ik};\ k = 1,\dots,i-1,i+1,\dots,N;\\
\pi^i_{k,N+1} &= A_{ik}^\top Y_i^\top - A_{ik}^\top Z_i;\ k = 1,\dots,i-1,i+1,\dots,N;\\
\pi^i_{1,k} &= A_{i1}^\top Z_i A_{ik};\ k = 2,\dots,i-1,i+1,\dots,N;\\
\pi^i_{N+1,N+1} &= P_i - Y_i - Y_i^\top;\\
\pi^i_{j,k} &= A_{ij}^\top Z_i A_{ik};\ j<k;\ j,k = 2,\dots,i-1,i+1,\dots,N;\\
\pi^i_{j,k} &= 0 \ \text{ on the other cases;}
\end{aligned}$$

$$g = \max_{i=\overline{1,N}} \left(\frac{\lambda_{\max}\left(E_i^\top P_i E_i\right)}{\lambda_{\min}\left(R_i\right)} \right) + \eta\ell\left(N-1\right) \max_{i=\overline{1,N}} \left(\frac{\lambda_{\max}\left(Q_i\right)}{\lambda_{\min}\left(R_i\right)} \right);\ \lambda = \max_{i=\overline{1,N}} \left(\frac{\lambda_{\max}\left(R_i\right)}{\lambda_{\min}\left(Q_i\right)} \right).$$

In the following theorem, we will present the regulary, causality and FTS of the system (1).

Theorem 1 (Finite-time stability) *Let* c_1, c_2, T *be positive numbers,* $R_i > 0$ *be given symmetric matrices for all* $i = \overline{1,N}$. *The large-scale system (1) is finite-time stable* $w.r.t$ (c_1, c_2, T, R) *if there are matrices* X_i, Z_i, *symmetric matrices* $P_i, Y_i, Q_i > 0, i = \overline{1,N}$, *and a scalar* $\eta > 0$ *such that*

$$\Upsilon_i = \begin{bmatrix}
\pi^i_{i,i} & \pi^i_{i,1} & \cdots & \pi^i_{i,i-1} & \pi^i_{i,i+1} & \cdots & \pi^i_{i,N} & \pi^i_{i,N+1}\\
* & \pi^i_{1,1} & \cdots & \pi^i_{1,i-1} & \pi^i_{1,i+1} & \cdots & \pi^i_{1,N} & \pi^i_{1,N+1}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots\\
* & * & \cdots & \pi^i_{i-1,i-1} & \pi^i_{i-1,i+1} & \cdots & \pi^i_{i-1,N} & \pi^i_{i-1,N+1}\\
* & * & \cdots & * & \pi^i_{i+1,i+1} & \cdots & \pi^i_{i+1,N} & \pi^i_{i+1,N+1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
* & * & \cdots & * & * & \cdots & * & \pi^i_{N+1,N+1}
\end{bmatrix} < 0, \tag{2}$$

$$\lambda(1+\eta)^{T+1} g c_1 \le c_2. \tag{3}$$

Proof. First, we prove the regularity and causality of the system. From LMI (2) we obtain that

$$\begin{bmatrix} \pi^i_{i,i} & \pi^i_{i,N+1} \\ * & \pi^i_{N+1,N+1} \end{bmatrix} < 0. \tag{4}$$

Since $\operatorname{rank} E_i = r_i < n_i$, from [18], there exists the matrices $N_i \in \mathbb{R}^{n_i \times (n_i - r_i)}$ satisfying $N_i^T E_i = 0$ and $\operatorname{rank} N_i = n_i - r_i$ for all $i = \overline{1,N}$. Leting

$$\overline{E}_i = \begin{bmatrix} 0 & 0 \\ E_i & 0 \end{bmatrix}; \qquad \overline{A}_i = \begin{bmatrix} E_i & I \\ -(A_i - E_i) & I \end{bmatrix}; \qquad \overline{P}_i = \begin{bmatrix} 0 & 0 \\ 0 & P_i \end{bmatrix};$$

$$\overline{Q}_i = \begin{bmatrix} \eta(N-1)Q_i & 0 \\ 0 & 0 \end{bmatrix}; \qquad \overline{N}_i = \begin{bmatrix} N_i & 0 \\ 0 & I \end{bmatrix}; \qquad \overline{X}_i = \begin{bmatrix} 0 & X_i^\top \\ 0 & -Y_i^\top \end{bmatrix};$$

Then, (4) is equivalent to

$$\Psi_i = \overline{X}_i \overline{N}_i^T \overline{A}_i + \overline{A}_i^T \overline{N}_i \overline{X}_i^T + \overline{Q}_i + \overline{E}_i^T \overline{P}_i + \overline{P}_i \overline{E}_i + \overline{P}_i < 0.$$

We can see that $\overline{P}_i \geq 0$, which gives

$$\overline{X}_i \overline{N}_i^T \overline{A}_i + \overline{A}_i^T \overline{N}_i \overline{X}_i^T + \overline{E}_i^T \overline{P}_i + \overline{P}_i \overline{E}_i < 0. \tag{5}$$

Since $\operatorname{rank}\overline{E}_i = \operatorname{rank}E_i = r_i < n_i$, there exist nonsingular matrices M_i and G_i satisfying $M_i \overline{E}_i G_i = \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}$. And since $N_i^T E_i = 0$ we have $\overline{N}_i^T \overline{E}_i = \begin{bmatrix} N_i^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_i & 0 \\ 0 & 0 \end{bmatrix} = 0$ or

$\overline{N}_i^T M_i^{-1} M_i \overline{E}_i G_i = \overline{N}_i^T M_i^{-1} \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix} = 0$. So, we can set

$$M_i \overline{A}_i G_i = \begin{bmatrix} \overline{A}_{11}^i & \overline{A}_{12}^i \\ \overline{A}_{21}^i & \overline{A}_{22}^i \end{bmatrix}, \quad M_i^{-\top} \overline{P}_i G_i = \begin{bmatrix} \overline{P}_{11}^i & \overline{P}_{12}^i \\ \overline{P}_{21}^i & \overline{P}_{22}^i \end{bmatrix}; G_i^\top \overline{X}_i = \begin{bmatrix} \overline{X}_1^i \\ \overline{X}_2^i \end{bmatrix}; M_i^{-T} \overline{N}_i = \begin{bmatrix} 0 \\ I \end{bmatrix} K_i,$$

where K_i are appropriately dimensioned nonsingular matrices. Multiplying by $G_i^\top$ and G_i on the left and on the right of LMI (5), repectively, we obtain

$$\Omega_i = \begin{bmatrix} \Omega_{1,1}^i & \overline{X}_1^i K_i^\top \overline{A}_{22}^i + \left(\overline{A}_{21}^i\right)^\top K_i \left(\overline{X}_2^i\right)^\top + \overline{P}_{12}^i \\ * & \overline{X}_2^i K_i^\top \overline{A}_{22}^i + \left(\overline{A}_{22}^i\right)^\top K_i \left(\overline{X}_2^i\right)^\top \end{bmatrix} < 0.$$

Applying the Schur complement lemma, we obtain $\overline{X}_2^i K_i^\top \overline{A}_{22}^i + \left(\overline{A}_{22}^i\right)^\top K_i \left(\overline{X}_2^i\right)^\top < 0,$ which gives $\det\left(\overline{A}_{22}^i\right) \neq 0$ for all $i = \overline{1, N}$. Hence, pair $\left(\overline{E}_i, \overline{A}_i\right)$, by [2, 18], is regular and causal. Furthermore, we know that $\det(sE_i - A_i) = \det(s\overline{E}_i - \overline{A}_i)$. Hence, $\det(sE_i - A_i)$ is not identical zero, or system (1) is regular and causal.

Next, we will show that large-scale system (1) is stable. To the end, we propose the following

Lyapunov function: $V(k,x_k)=\sum_{i=1}^{N}\left[V_{i1}(k,x_k)+V_{i2}(k,x_k)\right]$, where

$$V_{i1}(k,x_k)=x_i(k)^\top E_i^\top P_i E_i x_i(k),\; V_{i2}(k,x_k)=\eta\sum_{j=1,j\neq i}^{N}\sum_{l=k-\ell_{ij}}^{k-1}x_j(l)^\top Q_j x_j(l).$$

Taking the difference variation of $V(k,x_k)$, we have

$$\begin{aligned}\Delta V_{i1}(k,x_k)&=V_{i1}(k+1,x_{k+1})-V_{i1}(k,x_k)\\ &=\left[E_i x_i(k+1)\right]^\top P_i\left[E_i x_i(k+1)\right]-x_i(k)^\top E_i^\top P_i E_i x_i(k)\\ &=\left[E_i x_i(k+1)-E_i x_i(k)\right]^\top P_i\left[E_i x_i(k+1)-E_i x_i(k)\right]\\ &\quad+2x_i(k)^\top E_i^\top P_i\left[E_i x_i(k+1)-E_i x_i(k)\right]\end{aligned}$$

$$\begin{aligned}\Delta V_{i2}(k,x_k)&=\eta\sum_{j=1,j\neq i}^{N}\sum_{l=k+1-\ell_{ij}}^{k}x_j(l)^\top Q_j x_j(l)-\eta\sum_{j=1,j\neq i}^{N}\sum_{l=k-\ell_{ij}}^{k-1}x_j(l)^\top Q_j x_j(l)\\ &=\eta\sum_{j=1,j\neq i}^{N}x_j(k)^\top Q_j x_j(k)-\eta\sum_{j=1,j\neq i}^{N}x_j(k-\ell_{ij})^\top Q_j x_j(k-\ell_{ij}),\end{aligned}$$

On the other hand, from (1), we can find some matrices X_i, Y_i, Z_i with appropriate dimension sattisfying the following equalities:

$$-2x_i(k)^\top X_i\left[(A_i-E_i)x_i(k)-\left[E_i x_i(k+1)-E_i x_i(k)\right]+\sum_{j=1,j\neq i}^{N}A_{ij}x_j(k-\ell_{ij})\right]=0;$$

$$2\left[E_i x_i(k+1)-E_i x_i(k)\right]^\top Y_i^\top\left[(A_i-E_i)x_i(k)-\left[E_i x_i(k+1)-E_i x_i(k)\right]+\sum_{j=1,j\neq i}^{N}A_{ij}x_j(k-\ell_{ij})\right]=0;$$

$$2\sum_{j=1,j\neq i}^{N}x_j(k-\ell_{ij})^\top A_{ij}^\top Z_i^\top\left[(A_i-E_i)x_i(k)-\left[E_i x_i(k+1)-E_i x_i(k)\right]+\sum_{j=1,j\neq i}^{N}A_{ij}x_j(k-\ell_{ij})\right]=0.$$

Furthermore,

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}x_j(k)^\top Q_j x_j(k)=(N-1)\sum_{i=1}^{N}x_i(k)^\top Q_i x_i(k).$$

Setting $\xi_i^\top=\Big[x_i(k)^\top\quad x_1^\top(k-\ell_{i1})\quad x_2^\top(k-\ell_{i2})\quad\cdots\quad x_{i-1}^\top(k-\ell_{i,i-1})\quad x_{i+1}^\top(k-\ell_{i,i+1})\cdots$

$\cdots\quad x_N^\top(k-\ell_{iN})\quad\left[E_i x_i(k+1)-E_i x_i(k)\right]^\top\Big]$, then we have

$$
\begin{aligned}
\Delta V(k,x_k) &= V(k+1,x_{k+1}) - V(k,x_k) = \sum_{i=1}^{N}\Big[\Delta V_{i1}(k,x_k) + \Delta V_{i2}(k,x_k)\Big] \\
&= \sum_{i=1}^{N}\Big[E_i x_i(k+1) - E_i x_i(k)\Big]^{\top} P_i \Big[E_i x_i(k+1) - E_i x_i(k)\Big] \\
&+ 2\sum_{i=1}^{N} x_i(k)^{\top} E_i^{\top} P_i \Big[E_i x_i(k+1) - E_i x_i(k)\Big] + \eta(N-1)\sum_{i=1}^{N} x_i(k)^{\top} Q_i x_i(k). \\
&- \eta\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} x_j(k-\ell_{ij})^{\top} Q_j x_j(k-\ell_{ij}) - 2\sum_{i=1}^{N} x_i(k)^{\top} X_i (A_i - E_i) x_i(k) \\
&+ 2\sum_{i=1}^{N} x_i(k)^{\top} X_i \Big[E_i x_i(k+1) - E_i x_i(k)\Big] - 2\sum_{i=1}^{N} x_i(k)^{\top} X_i \sum_{j=1,j\neq i}^{N} A_{ij} x_j(k-\ell_{ij}) \\
&+ 2\sum_{i=1}^{N}\Big[E_i x_i(k+1) - E_i x_i(k)\Big]^{\top} Y_i^{\top} (A_i - E_i) x_i(k) \\
&- 2\sum_{i=1}^{N}\Big[E_i x_i(k+1) - E_i x_i(k)\Big]^{\top} Y_i^{\top} \Big[E_i x_i(k+1) - E_i x_i(k)\Big] \\
&+ 2\sum_{i=1}^{N}\Big[E_i x_i(k+1) - E_i x_i(k)\Big]^{\top} Y_i^{\top} \sum_{j=1,j\neq i}^{N} A_{ij} x_j(k-\ell_{ij}) \\
&+ 2\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} x_j(k-\ell_{ij})^{\top} A_{ij}^{\top} Z_i^{\top} (A_i - E_i) x_i(k) \\
&- 2\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} x_j(k-\ell_{ij})^{\top} A_{ij}^{\top} Z_i^{\top} \Big[E_i x_i(k+1) - E_i x_i(k)\Big] \\
&+ 2\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} x_j(k-\ell_{ij})^{\top} A_{ij}^{\top} Z_i^{\top} \sum_{j=1,j\neq i}^{N} A_{ij} x_j(k-\ell_{ij}) \\
&\leq \sum_{i=1}^{N} \xi_i^{\top} \Theta_i \xi_i,
\end{aligned}
\tag{1.6}
$$

in which

$$
\Theta_i = \begin{bmatrix}
\chi_{i,i}^{i} & \chi_{i,1}^{i} & \cdots & \chi_{i,i-1}^{i} & \chi_{i,i+1}^{i} & \cdots & \chi_{i,N}^{i} & \chi_{i,N+1}^{i} \\
* & \chi_{1,1}^{i} & \cdots & \chi_{1,i-1}^{i} & \chi_{1,i+1}^{i} & \cdots & \chi_{1,N}^{i} & \chi_{1,N+1}^{i} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
* & * & \cdots & \chi_{i-1,i-1}^{i} & \chi_{i-1,i+1}^{i} & \cdots & \chi_{i-1,N}^{i} & \chi_{i-1,N+1}^{i} \\
* & * & \cdots & * & \chi_{i+1,i+1}^{i} & \cdots & \chi_{i+1,N}^{i} & \chi_{i+1,N+1}^{i} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & \cdots & * & * & \cdots & * & \chi_{N+1,N+1}^{i}
\end{bmatrix},
$$

$$
\begin{aligned}
\chi_{i,i}^{i} &= \eta\big(N-1\big) Q_i - X_i^{\top}(A_i - E_i) - (A_i - E_i)^{\top} X_i; \\
\chi_{i,k}^{i} &= -X_i A_{ik} + (A_i - E_i)^{\top} Z_i A_{ik};\ k = 1,\ldots,i-1,i+1,\ldots N; \\
\chi_{i,N+1}^{i} &= E_i^{\top} P_i + (A_i - E_i)^{\top} Y_i + X_i; \\
\chi_{k,k}^{i} &= -\eta Q_k + A_{ik}^{\top} Z_i A_{ik} + A_{ik}^{\top} Z_i^{\top} A_{ik}; k = 1,\ldots,i-1,i+1,\ldots N; \\
\chi_{k,N+1}^{i} &= A_{ik}^{\top} Y_i - A_{ik}^{\top} Z_i^{\top}; k = 1,\ldots,i-1,i+1,\ldots N;
\end{aligned}
$$

$$\chi^i_{N+1,N+1} = P_i - Y_i - Y_i^\top;$$
$$\chi^i_{j,k} = A_{ik}^\top Z_i^\top A_{ik}, j = 1,\dots,i-1,i+1,\dots,N; k = 2,\dots,i-1,i+1,\dots,N, j \neq k;$$
$$\chi^i_{j,k} = 0 \quad \text{on the other cases;}$$

The inequality (2) gives $\Theta_i < 0, i = \overline{1,N},$ and from (6), we obtain

$$\Delta V\left(k, x_k\right) \leq 0.$$

This implies

$$V\left(k+1, x_{k+1}\right) \leq V\left(k, x_k\right) \leq (1+\eta)^{k+1} V\left(0, x_0\right) \leq (1+\eta)^{T+1} V\left(0, x_0\right) \text{ for all } k \in [0, T],$$

where

$$\begin{aligned}
V\left(0, x_0\right) &= \sum_{i=1}^{N}\left[x_i^\top\left(0\right)E_i^\top P_i E_i x_i\left(0\right) + \eta \sum_{j=1, j\neq i}^{N} \sum_{l=-\ell_{ij}}^{-1} x_j^\top\left(l\right)Q_j x_j\left(l\right)\right] \\
&\leq \sum_{i=1}^{N}\left[\frac{\lambda_{\max}\left(E_i^\top P_i E_i\right)}{\lambda_{\min}\left(R_i\right)} x_i^\top\left(0\right)R_i x_i\left(0\right) + \eta \sum_{j=1, j\neq i}^{N} \sum_{l=-\ell_{ij}}^{-1} \frac{\lambda_{\max}\left(Q_j\right)}{\lambda_{\min}\left(R_j\right)} x_j^\top\left(l\right)R_j x_j\left(l\right)\right] \\
&\leq \max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(E_i^\top P_i E_i\right)}{\lambda_{\min}\left(R_i\right)}\right)\sum_{i=1}^{N} x_i^\top\left(0\right)R_i x_i\left(0\right) \\
&\quad + \eta \max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(Q_j\right)}{\lambda_{\min}\left(R_i\right)}\right)\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \ell_{ij} \sup_{\left[-\ell_{ij}\leq\theta\leq-1\right]} x_j^\top\left(\theta\right)R_j x_j(\theta) \\
&\leq \max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(E_i^\top P_i E_i\right)}{\lambda_{\min}\left(R_i\right)}\right)\sum_{i=1}^{N} x_i^\top\left(0\right)R_i x_i\left(0\right) \\
&\quad + \eta \max_{i=\overline{1,N}}\left(\frac{\lambda_{\max}\left(Q_j\right)}{\lambda_{\min}\left(R_i\right)}\right)\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \ell \sup_{\left[-\ell\leq\theta\leq0\right]} x_j^\top\left(\theta\right)R_j x_j(\theta) \\
&\leq \left[\max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(E_i^\top P_i E_i\right)}{\lambda_{\min}\left(R_i\right)}\right) + \eta\ell\left(N-1\right)\max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(Q_i\right)}{\lambda_{\min}\left(R_i\right)}\right)\right]c_1 =: \lambda_1 c_1.
\end{aligned}$$

So

$$\begin{aligned}
x^\top\left(k\right)Rx\left(k\right) = \sum_{i=1}^{N} x_i^\top\left(k\right)R_i x_i\left(k\right) &\leq \max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(R_i\right)}{\lambda_{\min}\left(Q_i\right)}\right)\sum_{i=1}^{N} x_i^\top\left(k\right)Q_i x_i\left(k\right) \\
&\leq \max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(R_i\right)}{\lambda_{\min}\left(Q_i\right)}\right)V\left(k+1, x_{k+1}\right) \\
&\leq \max_{i=\underline{1,N}}\left(\frac{\lambda_{\max}\left(R_i\right)}{\lambda_{\min}\left(Q_i\right)}\right)(1+\eta)^{T+1}\lambda_1 c_1 \leq c_2,
\end{aligned}$$

which completes the proof of the theorem.

Remark. It is notable that the proposed result is not true for $N = 1$ (normal singular discrete-time systems). In [8], the authors solved the suboptimal control problem of finite-time H_∞ control and guaranteed cost control for linear singular large-scale discrete-time systems with delays and bounded disturbance. In this paper, building upon the results in [18] by Wu et al. for general discrete time systems, we showed the finite-time stability of the system (1) without disturbance and control function.

4. Example

Now, we give an example which illustrate the proposed method. Consider the large-scale discrete-time system with with *N=3* includes 3 subsystems where

$$E_1 = \begin{pmatrix} -5 & 1 \\ 0 & 0 \end{pmatrix}; E_2 = \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix}; E_3 = \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}; A_1 = \begin{pmatrix} 3 & 0 \\ -1 & 3.8 \end{pmatrix}; A_2 = \begin{pmatrix} 3 & 1.8 \\ 1 & -2 \end{pmatrix}; A_3 = \begin{pmatrix} 3 & 1 \\ -1 & 4.1 \end{pmatrix};$$

$$A_{12} = \begin{pmatrix} 1.5 & 0.3 \\ 2 & 1 \end{pmatrix}; A_{13} = \begin{pmatrix} 1.5 & 1 \\ 0 & 1.1 \end{pmatrix}; A_{21} = \begin{pmatrix} 0.5 & 0.5 \\ -1.5 & -0.9 \end{pmatrix}; A_{23} = \begin{pmatrix} 1.1 & 1.2 \\ 0.4 & -1 \end{pmatrix}; A_{31} = \begin{pmatrix} 0.5 & 0.2 \\ 1 & 1 \end{pmatrix};$$

$$A_{32} = \begin{pmatrix} -1.5 & 0 \\ 5 & -0.9 \end{pmatrix}; R_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; R_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}; R_3 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

The delays include $\ell_{12} = 1; \ell_{13} = 3; \ell_{21} = 1; \ell_{23} = 2; \ell_{31} = 2; \ell_{32} = 3$ and exist the matrices $N_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}; N_2 = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}; N_3 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ satisfying $N_1^T E_1 = N_2^T E_2 = N_3^T E_3 = 0.$ If take $c_1 = 1;\ c_2 = 14.3;\ \eta = 0.1;\ T = 10$, by using Matlabs Control, we obtain the matrices solutions are

$$X_1 = \begin{pmatrix} 0.1623 & 0.0286 \\ -0.2192 & 0.2349 \end{pmatrix}; X_2 = \begin{pmatrix} 0.1843 & 0.1454 \\ 0.2645 & 0.0003 \end{pmatrix}; X_3 = \begin{pmatrix} 0.7214 & 0.4649 \\ -0.1404 & 0.0574 \end{pmatrix};$$

$$Y_1 = \begin{pmatrix} 0.0908 & -0.0656 \\ -0.0656 & 0.1421 \end{pmatrix}; Y_2 = \begin{pmatrix} 0.1143 & 0.0736 \\ 0.0736 & 0.1506 \end{pmatrix}; Y_3 = \begin{pmatrix} 0.3155 & 0.0200 \\ 0.0200 & 0.0869 \end{pmatrix};$$

$$Z_1 = \begin{pmatrix} -0.0209 & 0.0365 \\ 0.0375 & -0.0445 \end{pmatrix}; Z_2 = \begin{pmatrix} 0.0198 & 0.0197 \\ 0.0113 & 0.0440 \end{pmatrix}; Z_3 = \begin{pmatrix} -0.0275 & -0.0551 \\ 0.0446 & -0.0844 \end{pmatrix};$$

$$P_1 = \begin{pmatrix} 0.0571 & -0.0485 \\ -0.0485 & 0.0676 \end{pmatrix}; P_2 = \begin{pmatrix} 0.0390 & 0.0353 \\ 0.0353 & 0.0868 \end{pmatrix}; P_3 = \begin{pmatrix} 0.2578 & 0.0571 \\ 0.0571 & 0.0173 \end{pmatrix};$$

$$Q_1 = \begin{pmatrix} 5.9506 & 1.0527 \\ 1.0527 & 2.3184 \end{pmatrix}; Q_2 = \begin{pmatrix} 4.4053 & 1.8358 \\ 1.8358 & 3.1386 \end{pmatrix}; Q_3 = \begin{pmatrix} 3.2483 & -0.6900 \\ -0.6900 & 2.7042 \end{pmatrix};$$

5. Conclusions

This paper proposed some criteria for finite-time stability of singular large-scale discrete-time systems with interconnected delays. By using matrix transformations combining Lyapunov function method, some conditions are expressed as feasible linear matrices inequalities conditions. An example is given to demonstrate the validity of the proposed results.

Declaration of Competing Interest

The authors of paper declare that there is no conflict of interests.

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