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On the second Hilbert coefficients and Cohen-Macaulay rings

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Abstract

In this paper, we investigate the relationship between second Hilbert coeficients and the index of reducibility of parameter ideals. We give some characterazations of Cohen-Macaulay rings via the above invariants.

Keywords: Cohen-Macaulay ring; Approximatetly Cohen-Macaulay ring; Hilbert coefficient; Multiplicity.

1. Introduction

Hilbert coefficients of R with respect to I , such that Hanor realigy
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 of Mathematics, 1435, 18 Hoang Quoc Yiet, 1030 commutative Noetherian local ring of dimension *d*, where *m* is the maximal
primary ideal of *R*. It is well-known that there are integers $e_i(I, R)$, called the
R with respect to *I*, such that
 $\ell_R(R_{\binom{n+1}{I}}^{\ell}) = e_0(I,$

$$
\ell_R(\frac{R}{\ell+1}) = e_0(I,R) \binom{n+d}{d} - e_1(I,R) \binom{n+d-1}{d-1} + \dots + (-1)^d e_d(I,R)
$$

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 Abstract

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3]) as the **Abstract**
 Abstract
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reducibility of parameter ideals. We give some characterazations of Cohen-Macaulay ring; via the** ween second Hilbert coeficients and the index of
haracterazations of Cohen-Macaulay rings via the
tly Cohen-Macaulay ring; Hilbert coefficient;
cal ring of dimension d, where m is the maximal
-known that there are integer Chern number of R with respect to I .

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The multiplicity and the Chern number of R with respect to parameter ideals can be used to deduce a lot of information on the structure of some classes of rings/modules, such as Gorenstein rings by Nagata [9]; Cohen-Macaulay modules by L. Ghezzi et al., W.V. Vasconcelos in [6, 18]; Buchsbaum modules by J. Stuckrad and W. Vogel in [13]; generalized Cohen-Macaulay modules by S. Goto et al. in [7].

Let M be a finitely generated module over a Noetherian ring R. A proper submodule N of M is called an *irreducible submodule* if N can not be written as an intersection of two properly larger submodules of M. In 1921, E. Noether [10] showed that every submodule N of M can be expressed as a finite intersection of irreducible submodule. Futhermore, the number of irreducible components of an irredundant irreducible decomposition of N , which is independent of the choice of the decomposition, is called *the index of reducibility* of N. Now our motivation stems from the work of the results of H.L. Truong et al. (see [11, 12, 14]) which demonstrate that reveals an interesting correlation between the Hilbert coefficients and the index of reducibility. The aim of our present work is to continue this research direction. Concretely, we will give some of characterizations of a Cohen-Macaulay rings in terms of the second Hilbert coefficients and the index of reducibility with respect to parameter ideals.

2. On the second Hilbert coefficients and Cohen-Macaulay ring

In what follows, throughout this paper, let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension d, where m is the maximal ideal and $k = R/m$ is the residue field of R. Suppose that R is a homomorphic image of a Cohen-Macaulay local ring. Let M be a non-zero finitely generated R module of dimension s.

Let I be an ideal of R such that $M/_{IM}$ has the finite length. Samuel showed that there is a polynomial $P_I(M)$ of degree s with rational coefficients such that $\ell_R\left(\frac{M}{I^{n+1}M}\right) = P_I(M)$ with *n* large enough, where $\ell_R(N)$ denotes the length of an R-module N. This implies that there are integers $e_i(I, M)$, called the *Hilbert coefficients* of M with respect to I, such that

$$
\ell_{R}\left(M/_{I^{n+1}M}\right) = e_{0}(I,M)\binom{n+s}{s} - e_{1}(I,M)\binom{n+s-1}{s-1} + \dots + (-1)^{s}e_{s}(I,M) \tag{2.1}
$$

for all $n \gg 0$. The Hilbert coefficients are used to measure the complexity of the structure of the module M. We call the leading coefficient $e_0(I,M)$ the multiplicity of M with respect to I, and $e_1(I, M)$ is said to be the *Chern number* of M with respect to I.

Definition 2.1 (see [8]). The *depth* of M, denoted by depth (M) , is defined to be the greatest integer t such that $H_m^i(M) = (0)$ for all $i < t$. We say that M is a Cohen-Macaulay module if depth (M) = dim M, i.e., $H_m^i(M) = (0)$ for all $i <$ dim M. In particular, if R is a Cohen-Macaulay Rmodule then R is said to be a *Cohen-Macaulay ring*.

Definition 2.2 (see [2]). We say that M is a generalized Cohen- Macaulay module if $\ell_R\left(H^i_m(M)\right)<\infty$ for all $0\leq i\leq s-1$.

By the definitions, every Cohen-Macaulay module is a generalized Cohen- Macaulay module. The following assertion is due to $[8]$ which gives a sufficient and necessary condition for R to be a Cohen-Macaulay ring in terms of the equality (2.1). Here we include an independent proof. *HPU2. Nat. Sci. Tech.* **2023**, 2(2), 03-10

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$$
e_0(q, R) \binom{n+d}{d} \le \ell_R \binom{R}{q^{n+1}} \le \ell_R \binom{R}{q} \binom{n+d}{d}
$$

for all $n \geq 0$. The above inequalities become equalities if and only if R is a Cohen-Macaulay ring.

 $\ell_R(\frac{R}{q^{n+1}})$ follows from the fact that
 $\ell_R(\frac{R}{q^{n+1}}) = \sum_{k=1}^{n} \ell_R(\frac{q^{i}}{q^{i+1}}) \leq \sum_{k=1}^{n} \ell_R(\frac{R}{q^{k}}) \mu(q^{i})$

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V \text{Cohen-Macaulay module is a sufficient and necessary condition for } R \text{ to be a of the equality (2.1). Here we include an independent proof.\n
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P_{0}(q, R) \binom{n+d}{d} \leq \ell_{R}(R/q^{n+1}) \leq \ell_{R}(R/q) \binom{n+d}{d}
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e_{i}(q, R) = 0, \text{ for all } i = 1, ..., d.
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\ell_{R}(R/q^{n+1}) = \sum_{i=0}^{n} \ell_{R}(q^{i}/q^{i+1}) \leq \sum_{i=0}^{n} \ell_{R}(R/q) \mu(q^{i})
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= \ell_{R}(R/q) \binom{n+d}{d}.
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V \text{allity, we let } S = R[X_{1}, X_{2}, ..., X_{d}] \text{ is the polynomial ring and let } S. \text{ Let } f_{i} = X_{i} - a_{i}, 1 \leq \forall i \leq d \text{ and put } q = (f_{1}, f_{2}, ..., f_{d})S. \text{ Then}
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S. \text{ as } S = R[f, f_{1}, ..., f_{1}] \text{. We look at the homomorphism}
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ule is a generalized Cohen-Macaulay module.
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ter ideal of R. Then there are inequalities
 $\left(\frac{n+1}{2}\right) \leq \ell_R (R/q) \binom{n+d}{d}$
ies if and on n-Macaulay ring in terms of the equality (2.1). Here we include an independent proof.
 Theorem 2.3. Let $q = (a_1, a_2, ..., a_d)$ be a parameter ideal of R. Then there are inequalities
 $e_0(q, R)^{(n+d)} \le \ell_A (R/q^{n+1}) \le \ell_A (R/q)^{(n+d)}$
 Cohen-Macaulay ring in terms of the equality (2.1). Here we include an independent proof.

Theorem 2.3. Let $q = (a_i, a_1, ..., a_n)$ be a parameter ideal of R. Then there are inequalities
 $e_n(q, R) \begin{pmatrix} n+d \\ d \end{pmatrix} \leq \ell_A(\mathcal{H}) q^{n+1$ **Theorem 2.3.** Let $q = (a_1, a_1, ..., a_n)$ be a parameter ideal of R. Then there are inequalities
 $e_5(q, R) \binom{n+d}{d} \leq \ell_x(R_{q^{-1}}) \leq \ell_x(R_{q^{-1}}) \leq \ell_x(R_{q^{-1}}) \leq \ell_x(R_{q^{-1}})$

or all $n \geq 0$. The above inequalities ℓ_x conce equal $e_0(q, R) \binom{n+d}{d} \leq \ell_R(\frac{R}{q}q^{n+1}) \leq \ell_R(\frac{R}{q}q) \binom{n+d}{d}$

for all $n \geq 0$. The above inequalities become equalities q and only if R is a Cohen-Macaulay ring.

Five this is the case, we have $e_1(q, R) = 0$, for all $i =$ $\ell_R(R_{q^{n+1}}) \leq \ell_R(R_{q})\binom{n+d}{d}$
qualities if and only if R is a Cohen-Macaulay ring.
all $i = 1, ..., d$.
 ℓ generators of an ideal I of R. The upper bound for
 $\ell(\frac{q'}{q^{n+1}}) \leq \sum_{i=0}^{n} \ell_R(R_{q})\mu(q')$
 $\leq \sum_{i=0}^{n} \ell_R(R_{q})\$ $\epsilon_0(q, R) \begin{vmatrix} \frac{1}{d} & \frac{1}{d} \\ 1 & \frac{1}{d} \end{vmatrix} \leq \ell_3(\frac{R}{q}q^{-1}) \leq \ell_3(\frac{R}{q}q) \begin{vmatrix} \frac{1}{d} & \frac{1}{d} \\ 0 & 1 \end{vmatrix}$

When this is the case, we have $\epsilon_1(q, R) = 0$, for all $i = 1, ..., d$.

When this is the case, we have $\epsilon_1(q,$ for all $n \ge 0$. The above inequalities become equalities if and only if R is a Cohen-Macaulay ring.

When this is the case, we have $e_i(q, R) = 0$, for all $i = 1, ..., d$.

Proof. Let $\mu(I)$ denote the minimal number of generato $\ell_{R}(\ell_{q^{m+1}}) = \sum_{r=0}^{n} \ell_{R}(\ell_{q}^{d})_{q^{n+1}} \leq \sum_{r=0}^{n} \ell_{R}(\ell_{q}^{d})\mu(q)$
 $\leq \sum_{r=0}^{n} \ell_{R}(\ell_{q}^{d})\binom{d}{d-1}$

For the rest of inequality, we let $S = R[X_{1}, X_{2}, ..., X_{d}]$ is the polynomial ring and let
 $M = mS + (X_{1}, X_{2},$ $\ell_R(N_{q^{n+1}}) = \sum_{i=0}^{\ell} \ell_R(\frac{q}{q^{i+1}}) \leq \sum_{i=0}^{\ell} \ell_R(\frac{R}{q}) \mu(q^i)$
 $\leq \sum_{i=0}^{\ell} \ell_R(\frac{R}{q}) \left(\frac{d+i-1}{d-1}\right)$
 $= \ell_R(\frac{R}{q}) \left(\frac{n+d}{d}\right)$.

of inequality, we let $S = R[X_1, X_2, ..., X_d]$ is the polynomial ring and let
 $..., X_d$ For the rest of inequality, we let $S = R[X_1, X_2, ..., X_d]$ is the polynomial ring and let
 $\mathcal{M} = mS + (X_1, X_2, ..., X_d)$ in S. Let $f_i = X_i - a_{i,1} \le \forall i \le d$ and put $\mathfrak{q} = (f_1, f_2, ..., f_d)$ S. Then
 $f_1, f_2, ..., f_d$ is a regular sequence For the rest of inequality, we let $S = R[X_1, X_2, ..., X_d]$ is the polynomial ring and let $M = mS + (X_1, X_2, ..., X_d)$ in S. Let $f_1 = X_1 = a_1, 1 \leq i \leq i$ and put $q = (f_1, f_2, ..., f_d)S$. Then $f_1, f_2, ..., f_d$ is a regular sequence in S, as $S =$ 1 ring and let

..., f_d)S. Then

homomorphism

and extend φ to

f₁, f_2 , ..., f_d is a = X_i – a_i , $1 \le \forall i \le d$ and put $q = (f_1, f_2, ..., f_d)S$. Then

as $S = R[f_i, f_2, ..., f_d]$. We look at the homomorphism
 $i \le d$. Then $q = \text{Ker } \varphi$. We put $C = S_M$ and extend φ to
 $\frac{\varphi}{S}$

g identifications
 $X_2, ..., X_d$)] = Let $f_i = X_i - a_i, 1 \le \forall i \le d$ and put $\mathfrak{q} = (f_1, f_2, ..., f_d)$. Then

in S, as $S = R[f_1, f_2, ..., f_d]$. We look at the homomorphism

call $1 \le i \le d$. Then $\mathfrak{q} = \text{Ker} \ \mathfrak{g}$. We put $C = S_M$ and extend \mathfrak{q} to

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 $\begin{array}{c} C \\$ $S = R[f_1, f_2, ..., f_d]$. We look at the homomorphism
 $S = R[f_1, f_2, ..., f_d]$. We look at the homomorphism
 $\leq d$. Then $\mathfrak{q} = \text{Ker } \mathfrak{p}$. We put $C = S_{\mathcal{M}}$ and extend \mathfrak{q} to
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 $\mathfrak{$ in S. Let $f_i = X_i - a_i$, $1 \le \forall i \le d$ and put $q = (f_1, f_2, ..., f_d)$ S. Then

equence in S, as $S = R[f_1, f_2, ..., f_d]$. We look at the homomorphism
 $= a_i$ for all $1 \le i \le d$. Then $q = \text{Ker } q$. We put $C = S_M$ and extend φ to
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 $\bigcup_{$ Let $f_i = X_i - a_i, 1 \le \forall i \le d$ and put $\mathfrak{q} = (f_1, f_2, ..., f_d)S$. Then

in S, as $S = R[f_1, f_2, ..., f_d]$. We look at the homomorphism

r all $1 \le i \le d$. Then $\mathfrak{q} = \text{Ker } \varphi$. We put $C = S_M$ and extend φ to
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 i $f_i = X_i - a_i, 1 \le \forall i \le d$ and put $q = (f_1, f_2, ..., f_d)S$. Then

1 S, as $S = R[f_1, f_2, ..., f_d]$. We look at the homomorphism

11 ≤ i ≤ d. Then q = Ker φ. We put $C = S_M$ and extend φ to
 C
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$$
R / \mathfrak{q}^{n+1} = B / \Big[\mathfrak{q}^{n+1} + (X_1, X_2, \dots, X_d) \Big] = C / \Big[\mathfrak{q}^{n+1} C + (X_1, X_2, \dots, X_d) C \Big]
$$

is a system of parameters for $\frac{C}{q^{n+1}C}$. Let . Let

$$
\operatorname{Assh}(C/\mathfrak{q} C)=\{\mathfrak{p}\in\operatorname{Ass}(C/\mathfrak{q} C)\mid \dim R/\mathfrak{p}=\dim C/\mathfrak{q} C\}.
$$

regular sequence in C, we get

$$
E = \int_{i}^{1} x_{i} - \lambda_{i} - a_{i}, 1 \leq \sqrt{2} a
$$
 and put $q = (f_{1}, f_{2}, \ldots, f_{d})$. Then
\nsequence in *S*, as *S* = *R*[*f*₁, *f*₂, ..., *f*_d]. We look at the homomorphism
\n*i*) = *a*_i for all 1 ≤ *i* ≤ *d*. Then *q* = Ker φ. We put *C* = *S*_{*M*} and extend φ to
\n*R*.
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C \xrightarrow{\psi} R
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C \xrightarrow{\psi} R
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= B / [\varphi^{n+1} + (X_1, X_2, \ldots, X_d)] = C / [\varphi^{n+1}C + (X_1, X_2, \ldots, X_d)C]
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Z_2, ..., X_d
$$
 is a system of parameters for $C_{\varphi^{n+1}C}$. Let
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C / \varphi C = \{p \in \text{Ass}(C / \varphi C) | \dim R / p = \dim C / \varphi C\}.
$$

\n
$$
C / \varphi C = \{p \in \text{Ass}(C / \varphi C) | \dim R / p = \dim C / \varphi C\}.
$$

\n
$$
C_{R}(R / \varphi^{n+1}) = \ell_C \Big(C / [\varphi^{n+1}C + (X_1, X_2, \ldots, X_d)C]\Big)
$$

\n
$$
= \ell_{\varphi} \Big((X_1, X_2, \ldots, X_d)C, C / \varphi^{n+1}C\Big)
$$

\n
$$
= \sum_{p \in \text{Ass}h_C C / \varphi C} \ell_{\varphi} \Big(\varphi_{p} / \varphi^{n+1} C_p\Big) \cdot e_0((X_1, X_2, \ldots, X_d)C, C / \varphi)
$$

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$$
= \sum_{p \in \text{Assh}_C C/qC} {n+d \choose d} \ell_{C_p} (C_p/qC_p) \cdot e_0((X_1, X_2, \dots, X_d)C, C/p))
$$
\n
$$
= {n+d \choose d} \sum_{p \in \text{Assh}_C C/qC} \ell_{C_p} (C_p/qC_p) \cdot e_0((X_1, X_2, \dots, X_d)C, C/p)
$$
\n
$$
= {n+d \choose d} e_0((X_1, X_2, \dots, X_d)C, C/qC) \text{ (by the associative formula)}
$$
\n
$$
= {n+d \choose d} e_0((X_1, X_2, \dots, X_d)C, C/qC) \text{ (by the associative formula)}
$$
\n
$$
= {n+d \choose d} e_0(q, R)
$$
\nfor all $n \ge 0$. Let $n \ge 0$ be now a fixed integer. We then have\n
$$
\ell_R (R/q^{n+1}) = {n+d \choose d} e_0(q, R)
$$
\nif and only if\n
$$
\ell_C (C / [q^{n+1}C + (X_1, X_2, \dots, X_d)C]) = e_0((X_1, X_2, \dots, X_d)C, C/q^{n+1}C),
$$
\nwhich is equivalent to saying that $\mathcal{L}_{q^{n+1}C}$ is a Cohen-Macaulay local ring. Since $q^{n+1}C$ is a perfect
\nideal of C (recall that $q = (f_1, f_2, \dots, f_d)$ is generated by a S-regular sequence f_1, f_2, \dots, f_d), this
\ncondition is equivalent to saying that the local ring C is Cohen-Macaulay, which means our base ring
\nR is Cohen-Macaulay. The problem we are interested in now is we need the following notion that how to characterize
\nthe structure of M in terms of its Hilbert coefficients, such as investigating the Hilbert coefficients of

for all $n \geq 0$. Let $n \geq 0$ be now a fixed integer. We then have

$$
\ell_R(R/q^{n+1}) = \binom{n+d}{d} e_0(q,R)
$$

if and only if

$$
\ell_C\left(C/\left[\mathfrak{q}^{n+1}C+(X_1,X_2,\cdots,X_d)C\right]\right)=e_0((X_1,X_2,\cdots,X_d)C,C/\mathfrak{q}^{n+1}C),
$$

which is equivalent to saying that $\frac{C}{q^{n+1}C}$ is a Cohen-Macaulay local ring. Since $q^{n+1}C$ is a perfect mula)
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 $\mathfrak{q}^{n+1}C$ is a perfect
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ans our base ring
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ert coefficients of condition is equivalent to saying that the local ring C is Cohen-Macaulay, which means our base ring R is Cohen-Macaulay. \Box

The problem we are interested in now is we need the following notion that how to characterize the structure of M in terms of its Hilbert coefficients, such as investigating the Hilbert coefficients of M so that M is a Cohen-Macaulay module. For this purpose, we need the following notion. A proper submodule N of M is called *irreducible* if it can not be written as an intersection of two strictly larger submodules of M . For a submodule N of M , the number of irreducible components of an irredundant irreducible decomposition of N, which is independent of the choice of the decomposition, is called the for all $n \ge 0$. Let $n \ge 0$ be now a fixed integer. We then have
 $\ell_x(R/q^{n+1}) = {n+d \choose d} e_n(R, R)$

if and only if
 $\ell_C (C/\left[q^{n+1}C + (X_1, X_2, \dots, X_d)C\right]) = e_0((X_1, X_2, \dots, X_d)C, C/q^{n+1}C),$

which is equivalent to saying that C_{q^{n+ $\ell_R(R/q^{n+1}) = {n+d \choose d} e_0(q, R)$
 $\ell_1, X_2, \dots, X_d)C] = e_0((X_1, X_2, \dots, X_d)C, C/q^{n+1}C),$
 $C'_q{}^{n+1}C$ is a Cohen-Macaulay local ring. Since $q^{n+1}C$ is a perfect
 \ldots, \ldots, f_d is generated by a S-regular sequence f_1, f_2, \ldots, f_d M M R ^M ^M ^M $R^j(q^{n+1}) = {n+d \choose d} e_0(q, R)$
 \cdots , X_d) $C]$ = $e_0((X_1, X_2, \cdots, X_d)C, C/q^{n+1}C)$,
 C is a Cohen-Macaulay local ring. Since $q^{n+1}C$ is a perfect

is generated by a *S*-regular sequence $f_1, f_2, ..., f_d$, this

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o saying that $\left\langle \int_{q^{n+1}}^{\alpha} C$ is a Cohen-Macaulay local ring. Since $q^{n+1}C$ is a perfect

hat $q = (f_1, f_2, \dots, f_d)$ is generated by a S-regular sequence is a Cohen-Macaulay local ring. Since $q^{n+1}C$ is a perfect
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The problem we are interested in now is we need the following notion that how to characteriz

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the structure of *M* in terms of its Hilbert coefficients, such as investigating the H
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The problem we are interested in now is we need the following notion that M variatize

tructure of M in terms o *M* so that *M* is a Cohen-Macaulay module. For this purpose, we need the following notion. A proper
submodule *N* of *M* is called *iredated* ir it can not be written as an intersection of two strictly larger
submodules

$$
\mathrm{ir}_M(\mathfrak{q}) := \mathrm{ir}_M(\mathfrak{q}M) = \ell_R \left([\mathfrak{q}M :_M \mathfrak{m}] \middle/ \mathfrak{q}M \right).
$$

We further set $I(q, M) = \ell(M/qM) - e_0(q, M)$. Note that, by Theorem 2.3, $I(q, M)$ is always non-negative. Let

$$
I(M) = \sup_{q} \left\{ I(q, M) \right\},\,
$$

 $r_i(M) = \ell((0) :_{H^i(M)} m)$ for $i = 1,...,s$.

Cohen-Macaulay ring with $d = \dim R > 0$ and q a parameter ideal of R contained in m^n for $n \gg 0$.
Then the followings hold true.

HPU2. Nat. Sci. Tech. **2023**, 2(2), 03-10
\n1)
$$
(-1)^{i} e_{i}(q, R) = \begin{cases} h_{0}(R) & \text{if } i = d, \\ \sum_{j=1}^{d-i} {d-i-1 \choose j-1} h_{i}(R) & \text{if } 0 < i < d. \end{cases}
$$
\n2)
$$
\operatorname{ir}_{R}(q) = \sum_{i=0}^{d} {d \choose i} r_{i}(R).
$$

We next will provide some of characterizations of 2-dimensional rings in terms of their second Hilbert coefficients (e.g., Cohen-Macaulay). Note that, in the case where dim $R = 2$, the following is immediately followed from Lemma 2.4.

Remark 2.5. Suppose that R is a generalized Cohen-Macaulay ring of dimesion $d = 2$ and q is a parameter ideal of R contained in m^n ($n \gg 0$). Then there are equalities

$$
e_2(\mathfrak{q}, R) = h_0(R)
$$
 and $\text{ir}_R(\mathfrak{q}) = r_0(R) + 2r_1(R) + r_2(R)$.

An R-module M is said to be *unmixed* if for all $P \in Ass \widehat{M}$ then $\dim \widehat{R}_{\widehat{P}} = \dim M$. In general, the local cohomology modules is not necessary finitely generated. However, it is an affirmation under the unmixed condition. The following result plays a key role in the arguments used in this paper.

Lemma 2.6 (see [5, Lemma 3.1]). Suppose that M is unmixed with $s = \dim M \ge 2$. Then $H_{m}^{1}(M)$ is a finitely generated R-module.

Our first main result of this paper is stated as follows.

Theorem 2.7. Assume that R is a unmixed local ring of dimension 2. Then the following statements are equivalent.

1) R is Cohen-Macaulay.

2) For all parameter ideal q of R, we have

 $e_2(q, R) = i r_p(q) - r_2(R)$.

3) For some (equivalently, every) parameter ideal $q \subseteq \mathfrak{m}^n$ $(n \gg 0)$ of R, we have

$$
e_2(\mathfrak{q},R) \geq \text{ir}_R(\mathfrak{q}) - \mathfrak{r}_2(R).
$$

Proof. 1) \Rightarrow 2) Since R is a Cohen-Macaulay ring of dimension 2, $H_m^0(R) = (0)$. On the other hands, since R is a unmixed local ring, by Remark 2.2 we get

 $e_2(q, R) = i r_R(q) - r_2(R) = 0.$

 $2) \Rightarrow 3$) It is immediately apparent.

3) \Rightarrow 1) Since R is unmixed and dim R = 2, R is generalized Cohen-Macaulay and $h_0(R) = 0$. Thus, by Lemma 2.1, we have $e_2(\mathfrak{q}, R) = h_0(R)$ and $i\mathfrak{r}_R(\mathfrak{q}) = r_0(R) + 2r_1(R) + r_2(R)$. Therefore,

$$
h_0(R) = e_2(\mathfrak{q}, R) \ge \text{ir}_R(\mathfrak{q}) - \mathfrak{r}_2(R) = \mathfrak{r}_0(R) + 2\mathfrak{r}_1(R).
$$

Thus, $r_0(R) = r_1(R) = 0$, whence $H_m^0(R) = H_m^1(R) = (0)$. It means that R is Cohen-Macaulay.

Let us give an example where Theorem 2.7 no longer holds if the dimension d is greater than 2 and R is not unmixed.

Example 2.8. Let $S = k[[X, Y, Z, W]]$ be the formal power series ring over a field k. Put $R = \frac{S}{X}$ $(X, Y, Z) \bigcap (W)$. Then

1) dim $R = 3$ and R is not unmixed. Hence, R is not a Cohen-Macaulay ring.

2) for all parameter ideal q of R , we have

$$
e_2(\mathfrak{q},R) \geq \text{ir}_R(\mathfrak{q}) - \mathfrak{r}_3(R).
$$

Indeed,

1) It is standard to check that $\text{Ass}(R) = \{(W), (X, Y, Z)\}\$. Thus dim $R = 3$ and R is not unmixed. It follows that R is not a Cohen-Macaulay ring.

2) We put $A = \frac{S}{W}$ and $B = \frac{S}{X}$ Note that A is a Gorenstein ring with dim A = 3 and B is a Gorenstein ring with dim $B = 1$. Therefore from the exact sequence $0 \rightarrow B \rightarrow R \rightarrow A \rightarrow 0$, one has $H_m^0(R) = H_m^2(R) = 0$ so that $H_m^1(R) \cong H_m^1(B)$ and $H_m^3(R) \cong H_m^3(B)$. Since A and B are Gorenstein, $r_0(R) = r_0(R) = 0$ and $r_1(R) = r_3(R) = 1$. Thus, by [15, Theorem 1.1], we have $ir_k(q) - r_d(R) = 2 - 1 = 1$. On the other hand, from the exact sequence $0 \to B \to R \to A \to 0$ and A is Cohen-Macaulay, we have

$$
0 \to B / q^{n+1} B \to R / q^{n+1} \to A / q^{n+1} A \to 0
$$

for all integers $n \ge 0$. Therefore, we have $\ell_R(\frac{R}{q^{n+1}}) = \ell_R(\frac{B}{q^{n+1}}) + \ell_R(\frac{A}{q^{n+1}})$, and so

$$
e_2(q, R) = \ell_R(\frac{B}{qB}) \ge 1
$$

because A and B are Cohen-Macaulay. Thus $e_2(q, R) \geq \text{ir}_{R}(q) - \text{r}_{2}(R)$.

 Under the unmixed assumption, we do not know if Theorem 2.7 holds true for the higher dimension case. Let us give a result where if the unmixed condition is replaced by the approximately Cohen-Macaulay condition then the higher dimension case still works. Recall that a non-Cohen-Macaulay local ring (R, \mathfrak{m}) is called *approximately Cohen-Macaulay* if there is an element $a \in \mathfrak{m}$ such that R/a^nR is a Cohen-Macaulay ring of dimension $d-1$ for all $n \ge 1$. The notion of the approximately Cohen-Macaulay module was first introduced by S. Goto [4]. In [1], N.T. Cuong and D.T. Cuong introduced this notion for the module case. S. Goto used the local cohomology to characterize for these rings (see [12, Theorem 1.1]).

Lemma 2.9 ([4, Theorem 1.1]). Suppose that R is an approximately Cohen-Macaulay ring but not a Cohen-Macaulay ring. Then $H_m^i(R) = 0$ for all $i \neq d - 1, d$.

Our second main result of this paper is stated as follows.

Theorem 2.10. Assume that R is approximately Cohen-Macaulay with dim $R = d \ge 3$. Then the following statements are equivalent.

- 1) R is Cohen-Macaulay.
- 2) For all parameter ideal q of R, we have

$$
e_2(\mathfrak{q},R) = \mathrm{ir}_R(\mathfrak{q}) - \mathrm{r}_d(R).
$$

3) For some (equivalently every) parameter ideal $q \subseteq \mathfrak{m}^n$ $(n \gg 0)$ of R, we have

$$
e_2(\mathfrak{q},R) \geq \text{ir}_R(\mathfrak{q}) - \mathfrak{r}_d(R).
$$

Proof.

1) \Rightarrow 2) Since R is Cohen-Macaulay, by Theorem 2.3 and Lemma 2.4, we have

$$
e_2(\mathfrak{q}, R) = \text{ir}_R(\mathfrak{q}) - \text{r}_d(R) = 0.
$$

 $2) \implies 3$) It is immediately apparent.

3) \Rightarrow 1) By Corollary 2.8 in [17], there exists a Goto sequence $x_1, x_2, ..., x_d$ on R such that *HPU2. Nat. Set. Tech.* 2023, 2(2), $03-10$
 $e_2(\mathbf{q}, R) \geq i\mathbf{r}_k(\mathbf{q}) - \mathbf{r}_j(R)$.
 Proof.

1) \Rightarrow 2) Since *R* is Cohen-Macaulay, by Theorem 2.3 and Lemma 2.4, we have
 $e_2(\mathbf{q}, R) = i\mathbf{r}_8(\mathbf{q}) - \mathbf{r}_4(R) = 0$.
 $q_i = (x_1, x_2, ..., x_{d-i})$ and $R_i = \frac{R}{q_i}$, $1 \le \forall i \le d$. Then by [11, Fact 2.3], $x_1, x_2, ..., x_0$ is a distinguished system of parameter of R satisfying the following conditions:

i) x_j are a superficial element of $\bigwedge^R \mathfrak{q}_{j-1}$ R with respect to q, for all $1 \le j \le d-2$. ii) Ass $(C_i / q_i C_j) \subseteq$ Assh $(C_i / q_i C_i) \cup \{m\}$, for all $j = 1, ..., d - 2$. iii) $0:_{R_{i-1}} x_i = H_{m}^{0}(R_{i-1}).$

Now, we put $A = R_{d-2}$. Observe that the short exact sequence $0 \to R \xrightarrow{q_{d-3}} R \to R_{d-3} \to 0$ induces the long exact sequence $0 \to H_m^0(R) \to H_m^0(R) \to H_m^0(R_{d-3}) \to H_m^1(R)$. Since R is approximately Cohen-Macaulay of dimension $d \ge 3$, by Lemma 2.10, $H_m^0(R) = H_m^1(R) = 0$, whence $H_m^0(R_{d-3}) = 0$. Hence by [9, Proposition 22.6], we have

$$
e_2(\mathfrak{q}, R) = e_2(\mathfrak{q}, A) - \ell((0) :_{R_{d-3}} x_{d-2}) = e_2(\mathfrak{q}, A)
$$
\n(2.2)

On the other hand, since $q \subseteq \mathfrak{m}^n$ (n >> 0), by [11, Lemma 3.5] we have $r_2(A) \ge r_d(R)$. Therefore

$$
\operatorname{ir}_R(\mathfrak{q}) - r_d(A) \ge \operatorname{ir}_A(\mathfrak{q}) - r_2(A) \tag{2.3}
$$

Note that $H_m^0(A)$ is the unmixed component of A and $\mathcal{A}_{H_m^0(A)}$ $_{m}^{0}(A)$ is unmixed of dimension 2. Thus $A_{\mathrm{H}_{\mathfrak{m}}^{0}(A)}'$ i $_{\text{m}}^{0}(A)$ is generalized Cohen-Macaulay by Lemma 2.6. It yields that A is generalized Cohen-Macaulay. Therefore, by Lemma 2.4, we have $e_1(aA, A) = h_0(A)$ and $\text{ir}_{A}(\mathfrak{q}) - r_2(A) = r_0(A) + 2r_1(A)$. Since $e_2(\mathfrak{q}, R) \ge \text{ir}_{R}(\mathfrak{q}) - r_d(R)$, by (2.2) and (2.3), we have $h_0(A) = r_0(A) + 2r_1(A)$, whence $r_1(A) = 0$. Therefore, $A_{\mathrm{H}^0_{\mathfrak{m}}}(A)$ i $_{m}^{0}(A)$ is Cohen-Macaulay. Hence R is Cohen-Macaulay (by Lemma [17, Lemma 3.2]). The proof therefore is completed.

3. Conclusions

 In this article, we provide a characterization the Cohen-Macaulay property of a 2-dimensional Noetherian local rings in terms of their second Hilbert coefficients. We also give necessary and sufficient conditions for approximately Cohen-Macaulay rings to be Cohen-Macaulay rings in terms of their second Hilbert coefficients.

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