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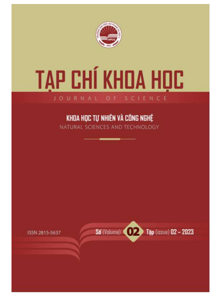

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On the second Hilbert coefficients and Cohen-Macaulay rings

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Abstract

In this paper, we investigate the relationship between second Hilbert coeficients and the index of reducibility of parameter ideals. We give some characterazations of Cohen-Macaulay rings via the above invariants.

Keywords: Cohen-Macaulay ring; Approximatetly Cohen-Macaulay ring; Hilbert coefficient; Multiplicity.

1. Introduction

Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring of dimension d, where $\mathfrak{m}$ is the maximal ideal. Let I be an $\mathfrak{m}$-primary ideal of R. It is well-known that there are integers $e_i(I,R)$, called the *Hilbert coefficients* of R with respect to I, such that

$$\ell_R(R/I^{n+1}) = e_0(I,R)\binom{n+d}{d} - e_1(I,R)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I,R)$$

for all $n \gg 0$. Here $\ell_R(N)$ denotes the length of an R-module N. The leading coefficient $e_0(I,R)$ is called *the multiplicity* of R with respect to I, and $e_1(I,R)$ is named by W.V. Vasconcelos ([18]) as the *Chern number* of R with respect to I.

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The multiplicity and the Chern number of R with respect to parameter ideals can be used to deduce a lot of information on the structure of some classes of rings/modules, such as Gorenstein rings by Nagata [9]; Cohen-Macaulay modules by L. Ghezzi et al., W.V. Vasconcelos in [6, 18]; Buchsbaum modules by J. Stuckrad and W. Vogel in [13]; generalized Cohen-Macaulay modules by S. Goto et al. in [7].

Let M be a finitely generated module over a Noetherian ring R. A proper submodule N of M is called an *irreducible submodule* if N can not be written as an intersection of two properly larger submodules of M. In 1921, E. Noether [10] showed that every submodule N of M can be expressed as a finite intersection of irreducible submodule. Futhermore, the number of irreducible components of an irredundant irreducible decomposition of N, which is independent of the choice of the decomposition, is called *the index of reducibility* of N. Now our motivation stems from the work of the results of H.L. Truong et al. (see [11, 12, 14]) which demonstrate that reveals an interesting correlation between the Hilbert coefficients and the index of reducibility. The aim of our present work is to continue this research direction. Concretely, we will give some of characterizations of a Cohen-Macaulay rings in terms of the second Hilbert coefficients and the index of reducibility with respect to parameter ideals.

2. On the second Hilbert coefficients and Cohen-Macaulay ring

In what follows, throughout this paper, let $(R,\mathfrak{m},k)$ be a Noetherian local ring of dimension d, where $\mathfrak{m}$ is the maximal ideal and $k = R/\mathfrak{m}$ is the residue field of R. Suppose that R is a homomorphic image of a Cohen-Macaulay local ring. Let M be a non-zero finitely generated R-module of dimension s.

Let I be an ideal of R such that M/IM has the finite length. Samuel showed that there is a polynomial $P_I(M)$ of degree s with rational coefficients such that $\ell_R\left(M/I^{n+1}M\right) = P_I(M)$ with n large enough, where $\ell_R(N)$ denotes the length of an R-module N. This implies that there are integers $e_i(I,M)$, called the *Hilbert coefficients* of M with respect to I, such that

$$\ell_R\left(M/I^{n+1}M\right) = e_0(I,M)\binom{n+s}{s} - e_1(I,M)\binom{n+s-1}{s-1} + \cdots + (-1)^s e_s(I,M) \qquad (2.1)$$

for all $n \gg 0$. The Hilbert coefficients are used to measure the complexity of the structure of the module M. We call the leading coefficient $e_0(I,M)$ *the multiplicity* of M with respect to I, and $e_1(I,M)$ is said to be the *Chern number* of M with respect to I.

Definition 2.1 (see [8]). The *depth* of M, denoted by $\operatorname{depth}(M)$, is defined to be the greatest integer t such that $\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $i < t$. We say that M is a *Cohen-Macaulay module* if $\operatorname{depth}(M) = \dim M$, i.e., $\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $i < \dim M$. In particular, if R is a Cohen-Macaulay R-module then R is said to be a *Cohen-Macaulay ring*.

Definition 2.2 (see [2]). We say that M is a *generalized Cohen- Macaulay module* if $\ell_R\left(\mathrm{H}^i_{\mathfrak{m}}(M)\right) < \infty$ for all $0 \le i \le s-1$.

By the definitions, every Cohen-Macaulay module is a generalized Cohen- Macaulay module. The following assertion is due to [8] which gives a sufficient and necessary condition for R to be a Cohen-Macaulay ring in terms of the equality (2.1). Here we include an independent proof.

Theorem 2.3. *Let $q=(a_1,a_2,...,a_d)$ be a parameter ideal of R. Then there are inequalities*

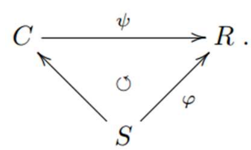
$$e_0(q,R)\binom{n+d}{d} \le \ell_R(R/q^{n+1}) \le \ell_R(R/q)\binom{n+d}{d}$$

for all $n \ge 0$. The above inequalities become equalities if and only if R is a Cohen-Macaulay ring. When this is the case, we have $e_i(q,R)=0$, for all $i=1,\ldots,d$.

Proof. Let $\mu(I)$ denote the minimal number of generators of an ideal I of R. The upper bound for $\ell_R(R/q^{n+1})$ follows from the fact that

$$\begin{aligned}\ell_R(R/q^{n+1}) &= \sum_{i=0}^{n}\ell_R(q^i/q^{i+1}) \le \sum_{i=0}^{n}\ell_R(R/q)\mu(q^i)\\ &\le \sum_{i=0}^{n}\ell_R(R/q)\binom{d+i-1}{d-1}\\ &= \ell_R(R/I)\binom{n+d}{d}.\end{aligned}$$

For the rest of inequality, we let $S=R[X_1,X_2,...,X_d]$ is the polynomial ring and let $\mathcal{M}=\mathfrak{m}S+(X_1,X_2,...,X_d)$ in S. Let $f_i=X_i-a_i, 1\le \forall i\le d$ and put $\mathfrak{q}=(f_1,f_2,...,f_d)S$. Then $f_1,f_2,...,f_d$ is a regular sequence in S, as $S=R[f_1,f_2,...,f_d]$. We look at the homomorphism $\varphi: S\to R$ defined by $\varphi(X_i)=a_i$ for all $1\le i\le d$. Then $\mathfrak{q}=\operatorname{Ker}\varphi$. We put $C=S_{\mathcal{M}}$ and extend φ to the homomorphism $\psi: C\to R$



Then $\operatorname{Ker}\psi=\mathfrak{q}C$ and we have the following identifications

$$R/\mathfrak{q}^{n+1}=B/\left[\mathfrak{q}^{n+1}+(X_1,X_2,\ldots,X_d)\right]=C/\left[\mathfrak{q}^{n+1}C+(X_1,X_2,\ldots,X_d)C\right]$$

for all $n\ge 0$, whence $X_1,X_2,...,X_d$ is a system of parameters for $C/\mathfrak{q}^{n+1}C$. Let

$$\operatorname{Assh}(C/\mathfrak{q}C)=\{\mathfrak{p}\in\operatorname{Ass}(C/\mathfrak{q}C)\mid \dim R/\mathfrak{p}=\dim C/\mathfrak{q}C\}.$$

Then, thanks to the associative formula of multiplicity together with the fact that $f_1,f_2,...,f_d$ is a regular sequence in C, we get

$$\begin{aligned}\ell_R(R/\mathfrak{q}^{n+1}) &= \ell_C\left(C/[\mathfrak{q}^{n+1}C+(X_1,X_2,\ldots,X_d)C]\right)\\ &\ge e_0\left((X_1,X_2,\ldots,X_d)C, C/\mathfrak{q}^{n+1}C\right)\\ &= \sum_{\mathfrak{p}\in\operatorname{Assh}_C C/\mathfrak{q}C}\ell_{C_\mathfrak{p}}\left(C_\mathfrak{p}/\mathfrak{q}^{n+1}C_\mathfrak{p}\right)\cdot e_0((X_1,X_2,\cdots,X_d)C, C/\mathfrak{p})\end{aligned}$$

$$= \sum_{\mathfrak{p}\in \mathrm{Assh}_C C/\mathfrak{q}C} \binom{n+d}{d} \ell_{C_\mathfrak{p}}(C_\mathfrak{p}/\mathfrak{q}C_\mathfrak{p}) \cdot e_0((X_1, X_2, \cdots, X_d)C, C/\mathfrak{p}))$$

$$= \binom{n+d}{d} \sum_{\mathfrak{p}\in \mathrm{Assh}_C C/\mathfrak{q}C} \ell_{C_\mathfrak{p}}(C_\mathfrak{p}/\mathfrak{q}C_\mathfrak{p}) \cdot e_0((X_1, X_2, \cdots, X_d)C, C/\mathfrak{p})$$

$$= \binom{n+d}{d} e_0((X_1, X_2, \cdots, X_d)C, C/\mathfrak{q}C) \text{ (by the associative formula)}$$

$$= \binom{n+d}{d} e_0(\mathfrak{q}, R)$$

for all $n \geq 0$. Let $n \geq 0$ be now a fixed integer. We then have

$$\ell_R(R/\mathfrak{q}^{n+1}) = \binom{n+d}{d} e_0(\mathfrak{q}, R)$$

if and only if

$$\ell_C\left(C/\left[\mathfrak{q}^{n+1}C + (X_1, X_2, \cdots, X_d)C\right]\right) = e_0((X_1, X_2, \cdots, X_d)C, C/\mathfrak{q}^{n+1}C),$$

which is equivalent to saying that $C/\mathfrak{q}^{n+1}C$ is a Cohen-Macaulay local ring. Since $\mathfrak{q}^{n+1}C$ is a perfect ideal of C (recall that $\mathfrak{q} = (f_1, f_2, ..., f_d)$ is generated by a S-regular sequence $f_1, f_2, ..., f_d$), this condition is equivalent to saying that the local ring C is Cohen-Macaulay, which means our base ring R is Cohen-Macaulay. □

The problem we are interested in now is we need the following notion that how to characterize the structure of M in terms of its Hilbert coefficients, such as investigating the Hilbert coefficients of M so that M is a Cohen-Macaulay module. For this purpose, we need the following notion. A proper submodule N of M is called *irreducible* if it can not be written as an intersection of two strictly larger submodules of M. For a submodule N of M, the number of irreducible components of an irredundant irreducible decomposition of N, which is independent of the choice of the decomposition, is called the *index of reducibility* of N and denoted by $\mathrm{ir}_M(N)$ (see [10]). For a parameter ideal $\mathfrak{q}$ for M, we have

$$\mathrm{ir}_M(\mathfrak{q}) := \mathrm{ir}_M(\mathfrak{q}M) = \ell_R\left([\mathfrak{q}M :_M \mathfrak{m}]/\mathfrak{q}M\right).$$

We further set $\mathrm{I}(\mathfrak{q}, M) = \ell(M/\mathfrak{q}M) - e_0(\mathfrak{q}, M)$. Note that, by Theorem 2.3, $\mathrm{I}(\mathfrak{q}, M)$ is always non-negative. Let

$$\mathrm{I}(M) = \sup_{\mathfrak{q}} \{\mathrm{I}(\mathfrak{q}, M)\},$$

where $\mathfrak{q}$ runs over all systems of parameters of M.

We summarize a way to compute the invariants introduced above. Let $h_i(M) = \ell_R(\mathrm{H}^i_\mathfrak{m}(M))$ and $r_i(M) = \ell\left((0) :_{\mathrm{H}^i_\mathfrak{m}(M)} \mathfrak{m}\right)$ for $i = 1, \ldots, s$.

Lemma 2.4 (see [9, Corollary 3.2] and [3, Theorem 1.1]). *Suppose that R is a generalized Cohen-Macaulay ring with $d = \dim R > 0$ and $\mathfrak{q}$ a parameter ideal of R contained in $\mathfrak{m}^n$ for $n \gg 0$. Then the followings hold true.*

1) $\left(-1\right)^i e_i(\mathfrak{q},R)=\begin{cases} h_0(R) & \text{if } i=d,\\ \sum_{j=1}^{d-i}\binom{d-i-1}{j-1}h_i(\mathrm{R}) & \text{if } 0<i<d.\end{cases}$

2) $\mathrm{ir}_R(\mathfrak{q})=\sum_{i=0}^{d}\binom{d}{i}r_i(R).$

We next will provide some of characterizations of 2-dimensional rings in terms of their second Hilbert coefficients (e.g., Cohen-Macaulay). Note that, in the case where $\dim R=2$, the following is immediately followed from Lemma 2.4.

Remark 2.5. Suppose that *R* is a generalized Cohen-Macaulay ring of dimesion $d=2$ and $\mathfrak{q}$ is a parameter ideal of *R* contained in $\mathfrak{m}^n$ $(n\gg 0)$. Then there are equalities

$$e_2(\mathfrak{q},R)=h_0(R) \text{ and } \mathrm{ir}_R(\mathfrak{q})=r_0(R)+2r_1(R)+r_2(R).$$

An *R*-module *M* is said to be *unmixed* if for all $\mathcal{P}\in \mathrm{Ass}\widehat{M}$ then $\dim \widehat{R}/\mathcal{P}=\dim M$. In general, the local cohomology modules is not necessary finitely generated. However, it is an affirmation under the unmixed condition. The following result plays a key role in the arguments used in this paper.

Lemma 2.6 (see [5, Lemma 3.1]). *Suppose that M is unmixed with* $s=\dim M\geq 2$. *Then* $\mathrm{H}^1_{\mathfrak{m}}(M)$ *is a finitely generated R-module.*

Our first main result of this paper is stated as follows.

Theorem 2.7. *Assume that R is a unmixed local ring of dimension 2. Then the following statements are equivalent.*

1) R is Cohen-Macaulay.

2) For all parameter ideal $\mathfrak{q}$ *of R, we have*

$e_2(\mathfrak{q},R)=\mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_2(R).$

3) For some (equivalently, every) parameter ideal $\mathfrak{q}\subseteq\mathfrak{m}^n$ $(n\gg 0)$ *of R, we have*

$$e_2(\mathfrak{q},R)\geq \mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_2(R).$$

Proof. $1)\Rightarrow 2)$ Since *R* is a Cohen-Macaulay ring of dimension *2*, $\mathrm{H}^0_{\mathfrak{m}}(R)=(0)$. On the other hands, since *R* is a unmixed local ring, by Remark 2.2 we get

$e_2(\mathfrak{q},R)=\mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_2(R)=0.$

$2)\Rightarrow 3)$ It is immediately apparent.

$3)\Rightarrow 1)$ Since R is unmixed and $\dim R=2$, R is generalized Cohen-Macaulay and $h_0(R)=0$. Thus, by Lemma 2.1, we have $e_2(\mathfrak{q},R)=h_0(R)$ and $\mathrm{ir}_R(\mathfrak{q})=r_0(R)+2r_1(R)+r_2(R)$. Therefore,

$$h_0(R)=e_2(\mathfrak{q},R)\geq \mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_2(R)=\mathrm{r}_0(R)+2\mathrm{r}_1(R).$$

Thus, $r_0(R)=r_1(R)=0$, whence $\mathrm{H}^0_{\mathfrak{m}}(R)=\mathrm{H}^1_{\mathfrak{m}}(R)=\left(0\right)$. It means that *R* is Cohen-Macaulay.

□

Let us give an example where Theorem 2.7 no longer holds if the dimension *d* is greater than 2 and *R* is not unmixed.

Example 2.8. Let $S=k[\![X,Y,Z,W]\!]$ be the formal power series ring over a field k. Put $R=S/(X,Y,Z)\cap(W)$. Then

1) $\dim R=3$ and *R* is not unmixed. Hence, *R* is not a Cohen-Macaulay ring.

2) for all parameter ideal $\mathfrak{q}$ of *R*, we have

$$e_2(\mathfrak{q},R)\geq \mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_3(R).$$

Indeed,

1) It is standard to check that $\mathrm{Ass}(R)=\{(W),\ (X,Y,Z)\}$. Thus $\dim R=3$ and *R* is not unmixed. It follows that *R* is not a Cohen-Macaulay ring.

2) We put $A=S/(W)$ and $B=S/(X,Y,Z)$. Note that *A* is a Gorenstein ring with $\dim A=3$ and *B* is a Gorenstein ring with $\dim B=1$. Therefore from the exact sequence $0\to B\to R\to A\to 0$, one has $\mathrm{H}^0_{\mathfrak{m}}(R)=\mathrm{H}^2_{\mathfrak{m}}(R)=0$ so that $\mathrm{H}^1_{\mathfrak{m}}(R)\cong \mathrm{H}^1_{\mathfrak{m}}(B)$ and $\mathrm{H}^3_{\mathfrak{m}}(R)\cong \mathrm{H}^3_{\mathfrak{m}}(B)$. Since *A* and *B* are Gorenstein, $\mathrm{r}_0(R)=\mathrm{r}_0(R)=0$ and $\mathrm{r}_1(R)=\mathrm{r}_3(R)=1$. Thus, by [15, Theorem 1.1], we have $\mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_d(R)=2-1=1$. On the other hand, from the exact sequence $0\to B\to R\to A\to 0$ and *A* is Cohen-Macaulay, we have

$$0\to B/q^{n+1}B\to R/q^{n+1}\to A/q^{n+1}A\to 0$$

for all integers $n\geq 0$. Therefore, we have $\ell_R(R/q^{n+1})=\ell_R(B/q^{n+1}B)+\ell_R(A/q^{n+1}A)$, and so

$$e_2(q,R)=\ell_R(B/qB)\geq 1$$

because *A* and *B* are Cohen-Macaulay. Thus $e_2(\mathfrak{q},R)\geq \mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_3(R)$.

Under the unmixed assumption, we do not know if Theorem 2.7 holds true for the higher dimension case. Let us give a result where if the unmixed condition is replaced by the approximately Cohen-Macaulay condition then the higher dimension case still works. Recall that a non-Cohen-Macaulay local ring $(R,\mathfrak{m})$ is called *approximately Cohen-Macaulay* if there is an element $a\in\mathfrak{m}$ such that R/a^nR is a Cohen-Macaulay ring of dimension $d-1$ for all $n\geq 1$. The notion of the approximately Cohen-Macaulay module was first introduced by S. Goto [4]. In [1], N.T. Cuong and D.T. Cuong introduced this notion for the module case. S. Goto used the local cohomology to characterize for these rings (see [12, Theorem 1.1]).

Lemma 2.9 ([4, Theorem 1.1]). Suppose that *R* is an approximately Cohen-Macaulay ring but not a Cohen-Macaulay ring. Then $\mathrm{H}^i_m(R)=0$ for all $i\neq d-1,d$.

Our second main result of this paper is stated as follows.

Theorem 2.10. *Assume that R is approximately Cohen-Macaulay with dim* $R=d\geq 3$. *Then the following statements are equivalent.*

1) R is Cohen-Macaulay.

2) For all parameter ideal $\mathfrak{q}$ *of R, we have*

$$e_2(\mathfrak{q},R)=\mathrm{ir}_R(\mathfrak{q})-\mathrm{r}_d(R).$$

3) For some (equivalently every) parameter ideal $\mathfrak{q}\subseteq\mathfrak{m}^n$ $(n\gg 0)$ *of R, we have*

$$e_2(\mathfrak{q},R) \geq \mathrm{ir}_R(\mathfrak{q}) - \mathrm{r}_d(R).$$

Proof.

$1) \Rightarrow 2)$ Since R is Cohen-Macaulay, by Theorem 2.3 and Lemma 2.4, we have

$$e_2(\mathfrak{q},R) = \mathrm{ir}_R(\mathfrak{q}) - \mathrm{r}_d(R) = 0.$$

$2) \Rightarrow 3)$ It is immediately apparent.

$3) \Rightarrow 1)$ By Corollary 2.8 in [17], there exists a Goto sequence $x_1, x_2, \ldots, x_d$ on R such that $\mathfrak{q} = (x_1, x_2, \ldots, x_d)$. Let $\mathfrak{q}_i = (x_1, x_2, \ldots, x_{d-i})$ and $R_i = R/\mathfrak{q}_i$, $1 \leq \forall i \leq d$. Then by [11, Fact 2.3], $x_1, x_2, \ldots, x_0$ is a distinguished system of parameter of R satisfying the following conditions:

i) x_j are a superficial element of $R/\mathfrak{q}_{j-1}$ with respect to $\mathfrak{q}$, for all $1 \leq j \leq d-2$.

ii) $\mathrm{Ass}(C_i/\mathfrak{q}_j C_i) \subseteq \mathrm{Assh}(C_i/\mathfrak{q}_j C_i) \cup \{\mathfrak{m}\}$, for all $j = 1, \ldots, d-2$.

iii) $0 :_{R_{i-1}} x_i = \mathrm{H}^0_{\mathfrak{m}}(R_{i-1})$.

Now, we put $A = R_{d-2}$. Observe that the short exact sequence $0 \to R \xrightarrow{q_{d-3}} R \to R_{d-3} \to 0$ induces the long exact sequence $0 \to \mathrm{H}^0_{\mathfrak{m}}(R) \to \mathrm{H}^0_{\mathfrak{m}}(R) \to \mathrm{H}^0_{\mathfrak{m}}(R_{d-3}) \to \mathrm{H}^1_{\mathfrak{m}}(R)$. Since R is approximately Cohen-Macaulay of dimension $d \geq 3$, by Lemma 2.10, $\mathrm{H}^0_{\mathfrak{m}}(R) = \mathrm{H}^1_{\mathfrak{m}}(R) = 0$, whence $\mathrm{H}^0_{\mathfrak{m}}(R_{d-3}) = 0$. Hence by [9, Proposition 22.6], we have

$$e_2(\mathfrak{q},R) = e_2(\mathfrak{q}A,A) - \ell((0) :_{R_{d-3}} x_{d-2}) = e_2(\mathfrak{q}A,A) \tag{2.2}$$

On the other hand, since $\mathfrak{q} \subseteq \mathfrak{m}^n (n >> 0)$, by [11, Lemma 3.5] we have $r_2(A) \geq r_d(R)$. Therefore

$$\mathrm{ir}_R(\mathfrak{q}) - r_d(A) \geq \mathrm{ir}_A(\mathfrak{q}) - r_2(A) \tag{2.3}$$

Note that $\mathrm{H}^0_{\mathfrak{m}}(A)$ is the unmixed component of A and $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is unmixed of dimension 2. Thus $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is generalized Cohen-Macaulay by Lemma 2.6. It yields that A is generalized Cohen-Macaulay. Therefore, by Lemma 2.4, we have $e_2(\mathfrak{q}A,A) = h_0(\mathrm{A})$ and $\mathrm{ir}_A(\mathfrak{q}) - r_2(A) = r_0(A) + 2r_1(A)$. Since $e_2(\mathfrak{q},R) \geq \mathrm{ir}_R(\mathfrak{q}) - \mathrm{r}_d(R)$, by (2.2) and (2.3), we have $h_0(A) = r_0(A) + 2r_1(A)$, whence $r_1(A) = 0$. Therefore, $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is Cohen-Macaulay. Hence R is Cohen-Macaulay (by Lemma [17, Lemma 3.2]). The proof therefore is completed.

□

3. Conclusions

In this article, we provide a characterization the Cohen-Macaulay property of a 2-dimensional Noetherian local rings in terms of their second Hilbert coefficients. We also give necessary and sufficient conditions for approximately Cohen-Macaulay rings to be Cohen-Macaulay rings in terms of their second Hilbert coefficients.

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