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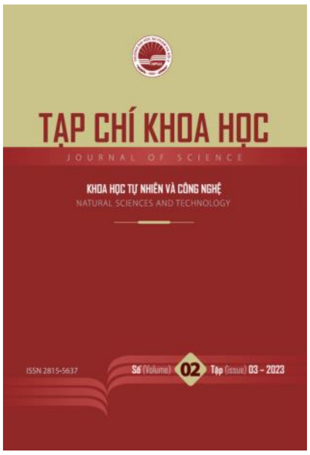

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Synchronization for fractional-order neural networks with unbounded delays

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Abstract

This paper deals with synchronization analysis problem for a class of fractional-order neural networks with unbounded delays. Using the Lyapunov function method combined with fractional Halanay inequality, we derive a novel sufficient condition for asymptotic stability of the error system resulting in two neural networks are synchronized. The obtained conditions are given in terms of linear matrix inequalities, which therefore can be efficiently checked. A numerical example is proposed to illustrate the effectiveness of the obtained results.

Keywords: Fractional order neural networks, synchronization analysis, unbounded delays, lyapunov function, asymptotic stable

1. Introduction

Neural networks have received the attention of many scientists in recent years due to its wide applications in image processing, combinatorial optimization, pattern recognition, adaptive control, and other areas [3, 13]. Theory of fractional calculus has been shown to be superior to classical differential and integral calculation in simulating materials and processes with memory [4, 9, 10, 11]. So, the neural networks model described by the fractional-order differential equation systems can describe the characteristics and properties of dynamical systems more efficiently and accurately. As a

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consequence, many important and interesting results on fractional-order neural networks have been reported and various issues have been studied by many authors [8, 14, 16].

Synchronization is a procedure in which two or more systems react with each other, leading to a joint development in some of their dynamic characteristics. The synchronization problem of integer order dynamic systems has received a lot of attention over the years [1, 5, 12]. However, the synchronization problem of fractional order neural networks is still limited. The reason is that the fractional order differential equations do not produce a semi-group operator. Therefore, we cannot easily extend the results of the synchronization problem for integer neural networks to fractional order neural networks.

This paper focuses on studying the synchronization analysis problem for a class of fractional order neural networks with unbounded delays by using Lyapunov functional method combined with fractional Halanay inequality and linear matrix inequality techniques. Compared with the previous work of fractional order neural networks [6], our result is more advantageous because the condition is given in the form of linear matrix inequalities, which can be effectively solved by various computational tools.

The remainder of this paper is organized as follows: Section 2 gives the main concepts and lemmas. Section 3 presents a synchronization scheme for fractional-order neural networks with unbounded delays. A numerical example is provided in Section 4 to illustrate the effectiveness of the proposed method.

Notations: $\mathbb{R}^{\boldsymbol{n}}$ and $\mathbb{R}^{\boldsymbol{n}\times\boldsymbol{m}}$ stand for the $\boldsymbol{n}$-dimensional vector space and real $(\boldsymbol{n}\times\boldsymbol{m})$ matrices, respectively. For any matrix $\boldsymbol{S}\in\mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}}, \boldsymbol{S}>\boldsymbol{0}(\boldsymbol{S}<\boldsymbol{0})$ means that it is positive-definite (negative-definite matrix) respectively, if $\boldsymbol{S}=\boldsymbol{S}^{\boldsymbol{T}}$ and $\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{S}\boldsymbol{x}>\boldsymbol{0}\ (\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{S}\boldsymbol{x}<\boldsymbol{0}), \forall\boldsymbol{x}\in\mathbb{R}^{\boldsymbol{n}}$. $\boldsymbol{S}^{\boldsymbol{T}}$ denotes the transposed matrix of $\boldsymbol{S}$. The symbol $*$ stands for symmetric block elements in a matrix. The operator $\mathbf{diag}\{.\}$ **represents a** diagonal matrix. $\mathbf{sym}(\boldsymbol{P})$ stands for $\boldsymbol{P}+\boldsymbol{P}^{\boldsymbol{T}}$, $\langle,\rangle$ **represents the inner product**.

2. Problem statement and preliminaries

Firstly, we introduce some concepts and properties of the fractional calculus, which are necessary for this present work.

Definition 1. ([9]) The Riemann-Liouville fractional integral and derivative of order $\alpha>0$ of a function $x(t)$ are defined as follows, respectively

$$I_t^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds, t\geq 0,$$

$${}_0^RD_t^{\alpha}x(t)=\frac{d^n}{dt^n}(I_t^{n-\alpha}x(t)), t\geq 0,$$

where $n=[\alpha]+1, \Gamma(.)$ is the gamma function, $\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt, s>0$.

Definition 2. ([9]) The Caputo derivative of order $\alpha>0$ is defined by

$${}_0^CD_t^{\alpha}x(t)= {}_0^RD_t^{\alpha}\left[x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k\right],$$

where $n=[\alpha]+1$.

Particularly, for $0<\alpha<1$,

$$ {}_0^C D_t^\alpha x(t) = {}_0^R D_t^\alpha [x(t) - x(0)]. $$

Next, the following useful properties of fractional calculus are given, which are used in this paper.

Property 1. ([9]) If $x(.) \in L_1[0, +\infty)$ and $0 < \alpha < 1$, then

$$ I_t^\alpha \left({}_0^C D_t^\alpha x(t) \right) = x(t) - x(0). $$

Property 2. ([9]) If $x(.) \in C[0, T]$, then we have

$$ I_t^{\alpha_1} \left(I_t^{\alpha_2} x(t) \right) = I_t^{\alpha_2} \left(I_t^{\alpha_1} x(t) \right) = I_t^{\alpha_1 + \alpha_2} (x(t)), \forall t \geq 0, \alpha_1, \alpha_2 > 0. $$

Consider the following two fractional-order neural networks

$$ \begin{cases} {}_0^C D_t^\alpha x(t) = -Ax(t) + Bf(x(t)) + Cg(x(t - \tau(t)), \\ p(t) = Hx(t), \end{cases} \tag{1} $$

$$ \begin{cases} {}_0^C D_t^\alpha y(t) = -Ay(t) + Bf(y(t)) + Cg(y(t - \tau(t)) + Wu(t), \\ q(t) = Hy(t), \end{cases} \tag{2} $$

where $\alpha \in (0,1)$ is the fractional order of the systems, $x(t) \in \mathbb{R}^n$ is the state vector of the system (1), $y(t) \in \mathbb{R}^n$ is the state vector of the system (2), $u(t) \in \mathbb{R}^k$ is the control vector, $p(t), q(t) \in \mathbb{R}^l$ are the output vectors, $f(.) = \left(f_1(.), \ldots, f_m(.)\right) \in \mathbb{R}^m, g(.) = (g_1(.), \ldots, g_m(.)) \in \mathbb{R}^m$ are activation functions, the timevarying function $\tau(t)$ satisfying $t - \tau(t) \geq -h$ for all $t \geq 0$ and $t - \tau(t) \to \infty$ as $t \to \infty. A \in \mathbb{R}^{n \times n}$ is the diagonal positive definite matrix, $B \in \mathbb{R}^{n \times m}$ is the connection weight matrix, $C \in \mathbb{R}^{n \times m}$ is the delayed connection weight matrix of the model and $W \in \mathbb{R}^{n \times k}, H \in \mathbb{R}^{l \times n}$ are known real constant matrices.

Assumption 1. The activation functions $f_i(.)$ and $g_i(.)$ $(i = 1,2, \ldots, m)$ are continuous, $f_i(0) = g_i(0) = 0$ and satisfy the following conditions on $\mathbb{R}$ for some known positive scalars $l_i, k_i(i = 1,2, \ldots, n)$

$$ l_i^- \leq \frac{f_i(a) - f_i(b)}{a - b} \leq l_i^+, \tag{3} $$

$$ t_i^- \leq \frac{g_i(a) - g_i(b)}{a - b} \leq t_i^+, \tag{4} $$

$\forall a, b \in \mathbb{R}, a \neq b.$

Set

$$ L_1 = \text{diag}\{l_1^-, l_2^-, \ldots, l_n^-\}, L_2 = \text{diag}\{l_1^+, l_2^+, \ldots, l_n^+\}, $$
$$ T_1 = \text{diag}\{t_1^-, t_2^-, \ldots, t_n^-\}, T_2 = \text{diag}\{t_1^+, t_2^+, \ldots, t_n^+\}. $$

We define the error as $e(t) = x(t) - y(t), z(t) = p(t) - q(t)$, and set $p(t) = f\left(x(t)\right) - f\left(y(t)\right)$, $\text{q}(t - \tau(t)) = g(x(t - \tau(t)) - g(y(t - \tau(t)$, the relevant error system can be formulated by

$$ \begin{cases} {}_0^C D_t^\alpha e(t) = -Ae(t) + Bp(t) + Cq(t - \tau(t)) - Wu(t), \\ z(t) = He(t). \end{cases} \tag{5} $$

With the control law $u(t) = Kz(t) = KHe(t)$, the corresponding closed-loop system of the system (5) is

$$ \begin{cases} {}_0^C D_t^\alpha e(t) = (-A - WKH)e(t) + Bp(t) + Cq(t - \tau(t)), \\ z(t) = He(t). \end{cases} \tag{6} $$

Definition 3. Systems (1) and (2) are said to be completely synchronized if the error vector $e(t)$

converges to zero, that is,

$$\lim_{t\to\infty} \| e(t) \| = \lim_{t\to\infty} \| x(t) - y(t) \| = 0.$$

Below, we recall several technical lemmas for deriving the main results.

Lemma 1. ([15]) Let $V: \mathbb{R}^n \to \mathbb{R}^+$be a convex and differentiable function on $\mathbb{R}^n$ such that $V(0) = 0$ and $\alpha \in (0,1), x(t) \in \mathbb{R}^n$ be a continuous function. We get

$${}_0^C D_t^\alpha V(x(t)) \le \left\langle \nabla V(x(t)),\ {}_0^C D_t^\alpha x(t) \right\rangle, t \ge 0$$

where $\nabla V(.)$ is the gradient of the function V and $\langle,\rangle$ is the inner product.

Lemma 2. (*Fractional-order Hanalay inequality*, [7]) Let $\alpha \in (0,1)$ and $V: [-h, \infty] \to \mathbb{R}^+$be continuous on $[0, \infty)$ and bounded on $[-h, 0]$. Asume that $\tau(.) \in C(\mathbb{R}^+, \mathbb{R}^+)$satisfies $\tau(t) \le t + h$ for some fixed $h > 0, t - \tau(t) \to \infty$ as $t \to \infty$. For some scalars $\lambda > \kappa > 0$, the following inequality holds

$${}_0^C D_t^\alpha V(t) \le -\lambda V(t) + \kappa \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma),$$

for all $t \ge 0$. Then

$$\lim_{t\to\infty} V(t) = 0.$$

Lemma 3. ([2]) Given $E \in \mathbb{R}^{p\times p}, G \in \mathbb{R}^{p\times q}, Z \in \mathbb{R}^{q\times p}, U \in \mathbb{R}^{q\times q}$ and scalar ζ. Inequality

$$E + Z^T G^T + GZ < 0.$$

is fulfilled if the following condition holds

$$\begin{bmatrix} E & \zeta G + Z^T U^T \\ * & -\zeta U - \zeta U^T \end{bmatrix} < 0.$$

3. The synchronization scheme for fractional-order neural networks with unbounded delays

The following theorem presents a sufficient condition for the error system (6) to be asymptotically stable resulting in the system (1) and the system (2) are synchronized.

Theorem 1. *The system (6) is asymptotically stable if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, two positive definite diagonal matrix $\Sigma, \Delta \in \mathbb{R}^{m\times m}$, a non-singular matrix $\Lambda \in \mathbb{R}^{n\times k}$, a matrix U and two positive numbers $\lambda > \kappa$ such that the following condition holds*

$$\begin{bmatrix} \Omega_{11} & 0 & PB + (L_1 + L_2)\Sigma & PC & \Omega_{15} \\ * & -\kappa P - 2T_1 \Delta T_2 & 0 & (T_1 + T_2)\Delta & 0 \\ * & * & -2\Sigma & 0 & 0 \\ * & * & * & -2\Delta & 0 \\ * & * & * & * & \Omega_{55} \end{bmatrix} < 0, \tag{7}$$

where

$$\begin{aligned} \Omega_{11} &= -PA - A^T P - 2L_1 \Sigma L_2 + \lambda P - WUH - H^T U^T W^T, \\ \Omega_{15} &= \zeta(W\Lambda - PW) + H^T U^T, \\ \Omega_{55} &= -\zeta U - \zeta U^T. \end{aligned}$$

In addition, the control gain matrix is given by $K = \Lambda^{-1} U$.

Proof. Consider the Lyapunov function candidate for the system (6)

$$V(e(t)) = e^T(t) P e(t)$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

Using Lemma 1, we calculate $\alpha(0<\alpha<1)$ Caputo derivative of $V(e(t))$ along the trajectories of system (6) as follows

$$\begin{aligned}
{}_0^C D_t^\alpha V\big(e(t)\big) &\le 2e^T(t)P\,{}_0^C D_t^\alpha e(t)\\
&= 2e^T(t)P\big[(-A-WKH)e(t)+Bp(t)+Cq\big(t-\tau(t)\big)\big]\\
&= e^T(t)(-PA-A^TP-PWKH-H^TK^TW^TP)e(t)\\
&+2e^T(t)PBp(t)\\
&+2e^T(t)PCq\big(t-\tau(t)\big). \qquad (8)
\end{aligned}$$

From (3), it is easy to derive that

$$[p^T(t)-e^T(t)L_1]\big[p(t)-L_2\,e(t)\big]\le 0. \qquad (9)$$

Therefore, for any positive definite diagonal matrix Σ, we always have

$$-2p^T(t)\Sigma p(t)+2e^T(t)(L_1+L_2)\Sigma p(t)-2e^T(t)L_1\Sigma L_2 e(t)\ge 0. \qquad (10)$$

Similarly, inequality (4) results

$$\begin{aligned}
&-2q^T\big(t-\tau(t)\big)\Delta q\big(t-\tau(t)\big)+2e^T\big(t-\tau(t)\big)(T_1+T_2)\Delta q\big(t-\tau(t)\big)\\
&-2e^T\big(t-\tau(t)\big)T_1\Delta T_2 e\big(t-\tau(t)\big)\ge 0, \qquad (11)
\end{aligned}$$

where diagonal matrix $\Delta>0$.

From conditions (8)-(11), we obtain

$$\begin{aligned}
{}_0^C D_t^\alpha V(e(t)) \quad &\le e^T(t)(-PA-A^TP-PWKH-H^TK^TW^TP)e(t)\\
&+2e^T(t)PBp(t)+2e^T(t)PCq\big(t-\tau(t)\big)-2p^T(t)\Sigma p(t)\\
&+2e^T(t)(L_1+L_2)\Sigma p(t)-2e^T(t)L_1\Sigma L_2 e(t)\\
&-2q^T\big(t-\tau(t)\big)\Delta q\big(t-\tau(t)\big)+2e^T\big(t-\tau(t)\big)(T_1+T_2)\Delta q\big(t-\tau(t)\big)\\
&-2e^T\big(t-\tau(t)\big)T_1\Delta T_2 e\big(t-\tau(t)\big)\\
&= e^T(t)(-PA-A^TP-PWKH-H^TK^TW^TP-2L_1\Sigma L_2)e(t)\\
&+2e^T(t)(PB+(L_1+L_2)\Sigma)p(t)+2e^T(t)PCq\big(t-\tau(t)\big)\\
&+2e^T\big(t-\tau(t)\big)(T_1+T_2)\Delta q\big(t-\tau(t)\big)\\
&-2p^T(t)\Sigma p(t)-2q^T\big(t-\tau(t)\big)\Delta q\big(t-\tau(t)\big)\\
&-2e^T\big(t-\tau(t)\big)T_1\Delta T_2 e\big(t-\tau(t)\big).
\end{aligned}$$

Hence, with constants $\lambda>\kappa>0$, we have

$$\begin{aligned}
&{}_0^C D_t^\alpha V\big(e(t)\big)+\lambda V\big(e(t)\big)-\kappa\sup_{-\tau(t)\le\sigma\le 0} V\big(e(t+\sigma)\big)\\
&\le e^T(t)(-PA-A^TP-PWKH-H^TK^TW^TP-2L_1\Sigma L_2)e(t)\\
&+2e^T(t)(PB+(L_1+L_2)\Sigma)p(t)+2e^T(t)PCq\big(t-\tau(t)\big)\\
&+2e^T\big(t-\tau(t)\big)(T_1+T_2)\Delta q\big(t-\tau(t)\big)\\
&-2p^T(t)\Sigma p(t)-2q^T\big(t-\tau(t)\big)\Delta q\big(t-\tau(t)\big)-2e^T\big(t-\tau(t)\big)T_1\Delta T_2 e\big(t-\tau(t)\big)\\
&+\lambda e^T(t)Pe(t)-\kappa\sup_{-\tau(t)\le\sigma\le 0} e^T(t+\sigma)Pe(t+\sigma)\\
&\le e^T(t)(-PA-A^TP-PWKH-H^TK^TW^TP-2L_1\Sigma L_2+\lambda P)e(t)\\
&+2e^T(t)(PB+(L_1+L_2)\Sigma)p(t)+2e^T(t)PCq\big(t-\tau(t)\big)
\end{aligned}$$

$$+2e^T\big(t-\tau(t)\big)(T_1+T_2)\Delta q\big(t-\tau(t)\big)$$
$$-2p^T(t)\Sigma p(t)-2q^T\big(t-\tau(t)\big)\Delta q\big(t-\tau(t)\big)$$
$$+e^T\big(t-\tau(t)\big)(-\kappa P-2T_1\Delta T_2)e\big(t-\tau(t)\big)\le \eta^T(t)\Omega\eta(t),$$

where

$$\eta(t)=[e^T(t)\quad e^T(t-\tau(t))\quad p^T(t)\quad q^T(t-\tau(t))]^T,$$

$$\Omega=\begin{bmatrix}\bar{\Omega}_{11} & 0 & PB+(L_1+L_2)\Sigma & PC\\ * & -\kappa P-2T_1\Delta T_2 & 0 & (T_1+T_2)\Delta\\ * & * & -2\Sigma & 0\\ * & * & * & -2\Delta\end{bmatrix},$$

$$\bar{\Omega}_{11}=-PA-A^TP-PWKH-H^TK^TW^TP-2L_1\Sigma L_2+\lambda P.$$

To solve the problem we need to eliminate the nonlinearities related to the control gain K. Set $K=\Lambda^{-1}U$, where Λ is a non-singular matrix, we have

$$PWKH=PW\Lambda^{-1}UH=(PW-W\Lambda)\Lambda^{-1}UH+WUH$$

Then it is easy to see

$$\Omega=\begin{bmatrix}-PA-A^TP+2L_1\Sigma L_2+\lambda P & 0 & PB+(L_1+L_2)\Sigma & PC\\ * & -\kappa P-2T_1\Delta T_2 & 0 & (T_1+T_2)\Delta\\ * & * & -2\Sigma & 0\\ * & * & * & -2\Delta\end{bmatrix}$$

$$+\begin{bmatrix}-PWKH-H^TK^TW^TP & 0 & 0 & 0\\ * & 0 & 0 & 0\\ * & * & 0 & 0\\ * & * & * & 0\end{bmatrix}$$

$$=\begin{bmatrix}\Omega_{11} & 0 & PB+(L_1+L_2)\Sigma & PC\\ * & -\kappa P-2T_1\Delta T_2 & 0 & (T_1+T_2)\Delta\\ * & * & -2\Sigma & 0\\ * & * & * & -2\Delta\end{bmatrix}$$
$$+\,\mathrm{sym}\left(\begin{bmatrix}W\Lambda-PW\\ 0\\ 0\end{bmatrix}\Lambda^{-1}[UH\quad 0\quad 0]\right),$$

where $\Omega_{11}=-PA-A^TP-2L_1\Sigma L_2+\lambda P-WUH-H^TU^TW^T$.

Applying Lemma 3, the inequality $\Omega<0$ can be guaranteed by the following condition

$$\begin{bmatrix}\Omega_{11} & 0 & PB+(L_1+L_2)\Sigma & PC & \Omega_{15}\\ * & -\kappa P-2T_1\Delta T_2 & 0 & (T_1+T_2)\Delta & 0\\ * & * & -2\Sigma & 0 & 0\\ * & * & * & -2\Delta & 0\\ * & * & * & * & \Omega_{55}\end{bmatrix}<0,$$

where $\Omega_{15}=\zeta(W\Lambda-PW)+H^TU^T$, $\Omega_{55}=-\zeta U-\zeta U^T$

Therefore, the conditions (8) implies $\Omega<0$. Thus,

$${}_0^CD_t^\alpha V(e(t))+\lambda V(e(t))-\kappa\sup_{-\tau(t)\le\sigma\le 0}V(e(t+\sigma))\le 0$$

Based on Lemma 2, it implies that

$$\lim_{t\to+\infty} V(e(t)) = 0.$$

Then, the system (6) is asymptotically stable, which the system (1) and the system (2) are said to be completely synchronized.

Remark 1. In recent times, the synchronization issue of the chaotic systems has raised wide concerns from scientists around the world. However, these results are mainly studied for integer order dynamic systems [1, 5, 12]. For the fractional order neural networks, constructing a positive definite function and calculating its derivative is still difficult. Therefore, how to design synchronization controller for the system has been a challenge. This paper considers the synchronization scheme for fractional order neural networks with unbounded delays by using the lyapunov function method combined with fractional Halanay inequality and LMI techniques. This makes the key contribution of this paper.

Remark 2. Synchronization scheme of fractional order neural networks with unbounded delays has been studied by B.B. He and H.C. Zhou in 2021 [6]. However, unlike our work, studying the problem using the Lyapunov function method combined with the LMI technique, the authors in [6] have investigated the synchronization analysis of fractional order neural networks using Laplace transform. Compared with the above work, our result is more advantageous because the condition is given in the form of linear matrix inequalities. So we can easily solve numerically by using the MATLAB software.

4. Example

Consider the system (6) with parameters described as: $\alpha \in (0,1)$,

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 2 & -3 \end{pmatrix}, W = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix},$

$H = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

$p(t) = 0.5(|e(t)+1| - |e(t)-1|),$

$q(t-\tau(t)) = 0.5(|e(t-\tau(t))+1| - |e(t-\tau(t))-1|),$

It's easy to see that $L_1 = T_1 = 0, L_2 = T_2 = 1$.

Choosing $\lambda = 1, \kappa = 0.5, \zeta = 1$.

Using LMI Control Toolbox in MATLAB, the condition (7) in Theorem 1 is feasible with

$$P = \begin{pmatrix} 1.16 & -0.008 \\ -0.008 & 0.745 \end{pmatrix}, \Sigma = \begin{pmatrix} 2.004 & 0 \\ 0 & 2.02 \end{pmatrix}, \Delta = \begin{pmatrix} 0.6132 & 0 \\ 0 & 0.4029 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} -0.0016 & 0.6824 \\ 0.2017 & 0.5699 \end{pmatrix}, \mathrm{U} = \begin{pmatrix} 0.7956 & -0.3389 \\ -0.3389 & 0.7181 \end{pmatrix}.$$

According to Theorem 1, the error system (6) is asymptotically stable with the feedback controller gain

$\mathrm{K} = \begin{pmatrix} -4.9405 & 4.93 \\ 1.154 & -0.4848 \end{pmatrix}.$

Therefore, the system (1) and the system (2) are said to be completely synchronized.

5. Conclusions

This research has studied the synchronization analysis problem using an output feedback control for a class of fractional-order neural networks with unbounded delays. The fractional Halanay inequality in conjunction with the Lyapunov function approach has allowed for the determination of a sufficient condition to guarantee synchronization of the systems under consideration. Additionally, a example is provided to demonstrate the viability and efficacy of the suggested approach.

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