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Algebraic dependences of meromorphic mappings sharing few moving hyperplanes with truncated multiplicity

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Abstract

In this article, we will prove an algebraic dependence theorem for meromorphic mappings into a complex projective space sharing few moving hyperplanes with different truncated multiplicity. Moreover, we also consider the weaker condition:

“$\nu_{(f,a_i),\le k_i} \le \nu_{(g,a_i),\le k_i}$” instead of “$\nu_{(f,a_i),\le k_i} = \nu_{(g,a_i),\le k_i}$”

for some moving hyperplanes a_i among the given moving hyperplanes. In order to implement this, besides using the technique reported by S. D. Quang in (Two meromorphic mappings having the same inverse images of some moving hyperplanes with truncated multiplicity, *Rocky Mountain J. Math*., vol. 52, no. 1, pp. 263–273, 2022) we have to separate the 2n + 2 moving hyperplanes a_i from the given p+1 moving hyperplanes. After that, we count multiples of the intersection of the inverse images of the mappings f and g sharing these moving hyperplanes. Our result is an improvement of many previous results in this topic.

Keywords: Nevanlinna theory, algebraic dependence, meromorphic mapping, hyperplanes

1. Introduction

From the “four and five values” theorems of R. Nevanlinna [1], many authors generalized the above results to the case of meromorphic mappings sharing fixed hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Recently, through the utilization of novel second main theorems for moving hyperplanes with truncated counting functions, as introduced by authors such as M. Ru and S. Stoll [2], M. Ru and J. T. Wang [3], D. D. Thai and S. D. Quang [4]–[6],..., many researches into this topic concerning mappings sharing moving hyperplanes has been conducted intensively and these studies have been referenced in [7]–[18].

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Firstly, we recall the result of H. Fujimoto [19] in 1999. He showed that there exists a positive integer l_0 such that if two meromorphic mappings f and g have the same inverse images counted with multiplicity l_0 for $2n+2$ fixed hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ then the mapping $f \times g$ is algebraically degenerate. In 2022, S. D. Quang [12] extended above result for two mappings sharing moving hyperplanes with different multiplicities. The purpose of this paper is to extend the result of S. D. Quang in the case where these two mappings not only have the same inverse images counted different multiplicities but also consider the weaker condition "$\nu_{(f,a_i),\leq k_i} \leq \nu_{(g,a_i),\leq k_i}$" instead of "$\nu_{(f,a_i),\leq k_i} = \nu_{(g,a_i),\leq k_i}$" for some moving hyperplanes a_i among the given moving hyperplanes. Here, we denote by "$\nu_{\varphi,\leq k}$" the divisor of distinct zeros with multiplicities not exceeding k; if the zeros have multiplicities which are greater than k, their multiplicities just equal to k. Namely, we prove the following theorem.

Theorem 1.1.

Let f^1, f^2 be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $p\ (\geq 2n+1)$ and $\{k_i\}_{i=1}^{p+1}$ be positive integers or ∞ and let $\{a_i\}_{i=1}^{p+1}$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position which are slow with respect to f^1 and f^2 with $(f^1, a_i) \not\equiv 0, (f^2, a_i) \not\equiv 0$. Set

$$A_1 = 1 - \frac{3n^2 q(q-2)}{2} \sum_{i=1}^{p+1} \frac{1}{k_i},$$

$$A_2 = 1 - \frac{3n^2 q(q-2)}{2} \sum_{i=1}^{2n+2} \frac{1}{k_i},$$

where $q = \binom{p+1}{n+2}$. Assume that

If $\nu_{(f^1,a_i),\leq k_i} = \nu_{(f^2,a_i),\leq k_i},\ 1 \leq i \leq 2n+2,$ and $\nu_{(f^1,a_i),\leq k_i} \leq \nu_{(f^2,a_i),\leq k_i},\ 2n+2 < i \leq p+1.$

$$\min\{A_1, A_2\} \geq 0 \quad (1)$$

$$\max\{A_1, A_2\} > 0 \quad (2)$$

then the map $f^1 \times f^2$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate over $\mathscr{R}\{a_i\}_{i=1}^{p+1}$.

With the same assumption as Theorem 1.1, the following corollary is an improvement of S. D. Quang [12] in the special case where $p = 2n+1$.

Corollary 1.2

If $\sum_{i=1}^{2n+2} \frac{1}{k_i} < \frac{2}{3n^2 q(q-2)}$ then the map $f^1 \times f^2$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate over $\mathscr{R}\{a_i\}_{i=1}^{2n+2}$.

2. Basic notions and auxiliary results from Nevanlinna theory

Let φ be a non-zero meromorphic function on $\mathbb{C}^m$. Now, we will use the standard notation from R. Nevanlinna theory due to [9], [13]. As usual, we denote by $N_{\varphi}^{[M]}(r)$, $N_{\varphi,\leq k}^{[M]}(r)$, $N_{\varphi,>k}^{[M]}(r)$ the counting functions of the divisors $\nu_{\varphi}, \nu_{\varphi,\leq k}, \nu_{\varphi,>k}$ respectively, and we denote by $T(r,\varphi)$ the characteristic function, $m(r,\varphi)$ the proximity function of φ, where φ is a meromorphic function on $\mathbb{C}^m$. For brevity we will omit the superscript $^{[M]}$ if $M = \infty$.

Let $f:\mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0:...:w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $f = (f_0, \dots, f_n)$ which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z):...:f_n(z))$ outside the analytic set $I(f) = \{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.

Throughout this paper, by the notation "$\|$ P" we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

Proposition 2.1

Let f be a nonzero meromorphic function on $\mathbb{C}^m$. Then

$$\|\quad m\left(r, \frac{\mathscr{D}^\alpha(f)}{f}\right) = O(\log^+ T(r,f))\ (\alpha \in \mathbb{Z}_+^m).$$

Let $a_1,...,a_q$ $(q \geq n+1)$ be q meromorphic mappings of $\mathbb{C}^m$ into the dual space $\mathbb{P}^n(\mathbb{C})^*$with reduced representations $a_i = (a_{i0}, \dots, a_{in})$ $(1 \leq i \leq q)$ We say that $a_1,...,a_q$are located in general position if $\det(a_{i_k l}) \not\equiv 0$ for any $1 \leq i_0 < i_1 < \cdots < i_n \leq q$. Let $\mathscr{M}_m$ be the field of all meromorphic functions on $\mathbb{C}^m$. Denote by $\mathscr{R}\{a_i\}_{i=1}^q \subset \mathscr{M}_m$ the smallest subfield which contains $\mathbb{C}$ and all $\frac{a_{ik}}{a_{il}}$ with $a_{il} \not\equiv 0$. Throughout this paper, if without any notification, the notation $\mathscr{R}$ always stands for $\mathscr{R}\{a_i\}_{i=1}^q$.

We call each meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ a moving hyperplane in $\mathbb{P}^n(\mathbb{C})$. A moving hyperplane a in $\mathbb{P}^n(\mathbb{C})$ is said to be "slow" (with respect to f) if $\|\ T_a(r) = o(T_f(r))$.

Let N be a positive integer and let V be a projective subvariety of $\mathbb{P}^N(\mathbb{C})$. Take a homogeneous coordinates $(\omega_0:\ldots:\omega_N)$ of $\mathbb{P}^N(\mathbb{C})$. Let F be a meromorphic mapping of $\mathbb{C}^m$ into V with a representation $F = (F_0, \dots, F_n)$.

Definition 2.2

The meromorphic mapping F is said to be algebraically degenerate over a subfield $\mathscr{R}$ of $\mathscr{M}_m$ if there exists a homogeneous polynomial $Q \in \mathscr{R}[\omega_0,\ldots,\omega_N]$ with the form

$$Q(z)(\omega_0,\ldots,\omega_N) = \sum_{I \in \mathscr{I}_d} a_I(z)\omega^I,$$

where d is an integer, $\mathscr{I}_d = \{(i_0,\ldots,i_N);\ 0 \leq i_j \leq d,\ \sum_{j=0}^N i_j = d\}, a_I \in \mathscr{R}$ and $\omega^I = \omega_0^{i_0}\ldots\omega_N^{i_N}$ for $I = (i_0,\ldots,i_N)$, such that

(i) $Q(z)(F_0(z),\ldots,F_N(z)) \equiv 0$ on $\mathbb{C}^m$,

(ii) there $\exists\ z_0 \in \mathbb{C}^m$ with $Q(z)(F_0(z),\ldots,F_N(z)) \not\equiv 0$ on V.

Let f and g be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with representations $f = (f_0, \dots, f_n)$ and $g = (g_0, \dots, g_n)$.

We consider $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ as a projective subvariety of $\mathbb{P}^{(n+1)^2-1}(\mathbb{C})$ by Segre embedding. Then the map $f \times g$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate over a subfield $\mathscr{R}$ of $\mathscr{M}_m$ if there exists a nontrivial polynomial

$$Q(z)(\omega_0,...,\omega_n,\omega_0',...,\omega_n') = \sum_{\substack{I=(i_0,...,i_n)\in\mathbb{Z}_+^{n+1}\\ i_0+\cdots+i_n=d}} \sum_{\substack{J=(j_0,...,j_n)\in\mathbb{Z}_+^{n+1}\\ j_0+\cdots+j_n=d'}} a_{IJ}(z)\omega^I\omega'^J,$$

where d, d' are positive integers, $a_{IJ}\in\mathcal{R}$, such that

$$Q(z)(f_0(z),\ldots,f_n(z),g_0(z),\ldots,g_n(z))\equiv 0.$$

Let $a:\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C})^*$ be a meromorphic mapping. Suppose that a have reduced representation $(a_0,\ldots,a_n)$. We put $(f,a):=\sum_{i=0}^{n} f_i a_i$. The definition of this function depends on the choices of reduced representation a, but its divisor $\nu_{(f,a)}$ does not depend on that. Similarly, we define the proximity function $m_{f,a}(r)$ and the first main theorem for moving hyperplanes (see [20]) as follows.

$$T_f(r)+T_a(r)=m_{f,a}(r)+N_{(f,a)}(r).$$

Proposition 2.3 *(see* [5], *Theorem 1.3).*

Let $f:\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^{q}(q\geq(n-k)(k+1)+n+2)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in the general position such that $(f,a_i)\not\equiv 0$, $1\leq i\leq q$, where $\mathrm{rank}_{\mathcal{R}}(f)=k+1$. Then we have

$$||\frac{q}{k+2}T_f(r)\leq\sum_{i=1}^{n}N^{[k]}_{(f,a_i)}(r)+o(T_f(r))+O(\max_{1\leq i\leq q}T_{a_i}(r)).$$

Proposition 2.4 *(see* [3]*).*

Let $f=(f_0,\ldots,f_n)$ be a reduced representation of a meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Assume that f_{n+1} is a holomorphic function with $f_0+\ldots+f_n+f_{n+1}=0$. If $\sum_{i\in I}f_i\neq 0$ for all $I\subseteq\{0,\ldots,n+1\}$,then

$$||T_f(r)\leq\sum_{i=0}^{n+1}N^{[n]}_{f_i}(r)+o(T_f(r)).$$

Proposition 2.5 *(see* [20], *Theorem 5.2.29).*

Let f be a nonzero meromorphic function on $\mathbb{C}^m$ with a reduced representation $f=(f_0,\ldots,f_n)$. Suppose that $f_k\not\equiv 0$, then

$$T(r,\frac{f_j}{f_k})\leq T(r,f)\leq\sum_{j=0}^{n}T(r,\frac{f_j}{f_k})+O(1).$$

3. Proof of Theorem 1.1

Let $H_1,\ldots,H_{2n+1}$ be $2n+1$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position given by

$$H_i:x_{i0}w_0+\ldots+x_{in}w_n=0\ (1\leq i\leq 2n+1).$$

We consider the rational map Φ: $\mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})\to\mathbb{P}^{2n}(\mathbb{C})$as follows:

For $v=(v_0:v_1:\ldots:v_n),w=(w_0:w_1:\ldots:w_n)\in\mathbb{P}^n(\mathbb{C})$, we define the value $\Phi(v,w)=(u_1:\ldots:u_{2n+1})\in\mathbb{P}^{2n}(\mathbb{C})$ by

$$u_i = \frac{x_{i0}v_0 + \ldots + x_{in}v_n}{x_{i0}w_0 + \ldots + x_{in}w_n}.$$

Lemma 3.1 *(see* [19]*, Proposition 5.9).*

The map Φ is a birational map of $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ to $\mathbb{P}^{2n}(\mathbb{C})$. Let $a_1, \ldots, a_{2n+1}$ be $2n+1$ moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with reduced representations

$$a_i = (a_{i0}, \ldots, a_{in}) \ (1 \leq i \leq 2n+1)$$

Let f^1 and f^2 be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representations

$$f^1 = (f_0^1, \ldots, f_n^1) \ and \ f^2 = (f_0^2, \ldots, f_n^2)$$

Define $h_i = \frac{(f^1, a_i)}{(f^2, a_i)}$ $(1 \leq i \leq 2n+1)$ and $h_I = \prod_{i \in I} h_i$ for each subset I of {1,..., 2n + 1}. Set $\mathcal{J} = \{I = (i_1, \ldots, i_n);\ 1 \leq i_1 < \ldots < i_n \leq 2n+1\}$.

Lemma 3.2 *(see* [15]*).*

If there exist functions $A_I \in \mathcal{R}\{a_i\}_{i=1}^{2n+1}$ $(I \in \mathcal{J})$, not all zero, such that

$$\sum_{i \in \mathcal{J}} A_I h_I \equiv 0,$$

then the map $f^1 \times f^2$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate over $\mathcal{R}\{a_i\}_{i=1}^{2n+1}$.

Lemma 3.3 *(see* [15]*).*

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $a_1, \ldots a_{n+1}$ be moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Then for each regular point z_0 of the analytic subset $\bigcup_{i=1}^{n+1}\{z: (f, a_i)(z) = 0\}$ with $z_0 \notin I(f)$, we have

$$\min_{1 \leq i \leq n+1} \nu^0_{(f, a_i)}(z_0) \leq \nu^0_{det\Phi}(z_0),$$

where $I(f)$ denotes the indeterminacy set of f and Φ is the matrix $(a_{ij}; 1 \leq i \leq n+1, 0 \leq j \leq n)$.

Proof of Theorem 1.1.

By changing the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$ if necessary, we may assume that $a_{i0} \not\equiv 0$ for all $1 \leq i \leq p+1$. We set $\tilde{a}_{ij} = \frac{a_{ij}}{a_{i0}}$,

$$\tilde{a}_i = (\frac{a_{i0}}{a_{i0}}, \frac{a_{i1}}{a_{i0}}, \ldots, \frac{a_{in}}{a_{i0}}),\ (f^1, \tilde{a}_i) = \sum_{j=0}^{n} \tilde{a}_{ij} f_j^1 \ \text{and} \ (f^2, \tilde{a}_i) = \sum_{j=0}^{n} \tilde{a}_{ij} f_j^2.$$

We suppose contrarily that the map $f^1 \times f^2$ is algebraically non-degenerate over $\mathcal{R}\{a_i\}_{i=1}^{p+1}$. We set $h_i = \frac{(f^1, \tilde{a}_i)}{(f^2, \tilde{a}_i)}$ $(1 \leq i \leq 2n+2)$.

Then $\frac{h_i}{h_j} = \frac{(f^1,\tilde{a}_i)(f^2,\tilde{a}_j)}{(f^2,\tilde{a}_i)(f^1,\tilde{a}_j)}$ does not depend on the choice of representations of f^1 and f^2. Since $\sum_{j=0}^{n}\tilde{a}_{ij}f_j^1 - h_i\sum_{j=0}^{n}\tilde{a}_{ij}f_j^2 = 0\ (1 \le i \le 2n+2)$ it implies that

$$\phi := \det(\tilde{a}_{i0}, \ldots, \tilde{a}_{in}, \tilde{a}_{i0}h_i, \ldots, \tilde{a}_{in}h_i;\ 1 \le i \le 2n+2) = 0. \tag{3}$$

For each subset $I \subset \{1,2,\ldots,p+1\}$, put $h_I = \prod_{i \in I} h_i$. We denote

$\mathscr{I} = \{(i_1,\ldots,i_{n+1}); 1 \le i_1 < \ldots < i_{n+1} \le 2n+2\}$.

For each $I = (i_1,\ldots,i_{n+1}) \in \mathscr{I}$, define

$$A_I = (-1)^{\frac{(n+1)(n+2)}{2} + i_1 + \ldots + i_{n+1}} \det(\tilde{a}_{i_r l};\ 1 \le r \le n+1, 0 \le l \le n)\det(\tilde{a}_{j_s l};\ 1 \le s \le n+1, 0 \le l \le n),$$

where $J = (j_1,\ldots,j_{n+1}) \in \mathscr{I}$ such that $I \cup J = \{1,2,\ldots,2n+2\}$. We have

$$\sum_{I \in \mathscr{I}} A_I h_I = 0.$$

We take a partition of $\mathscr{I} = \mathscr{I}_1 \cup \ldots \cup \mathscr{I}_k$ with the following properties:

- $I_t \cap I_s = \varnothing$, $1 \le t < s \le k$.
- $\sum_{I \in \mathscr{I}_t} A_I h_I = 0$, $1 \le t \le k$.
- $\sum_{I \in \mathscr{J}} A_I h_I \ne 0$ for any proper subset $\mathscr{J}$ of $\mathscr{I}_t$, $1 \le t \le k$.

For each $1 \le t \le k$, we set $n_t = \sharp\mathscr{I}_t - 2$ and assume that $\mathscr{I}_t = \{I_{0t},\ldots,I_{(n_t+1)t}\}$. We denote by F_t the meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^{n_t}(\mathbb{C})$ with the presentation $(A_{I_{0t}}h_{I_{0t}} : \ldots : A_{I_{n_t t}}h_{I_{n_t t}})$. For each $1 \le i \le 2n+2$, we define $S(i)$ the set of all indices $j \ne i$ such that there exist $\mathscr{I}_t$, two elements $I, I' \in \mathscr{I}_t$ satisfying

$$\frac{h_I}{h_{I'}} = \frac{h_i}{h_j}.$$

Firstly, we will prove following Claim.

Claim. For each $1 \le i \le 2n+2$, $\sharp S(i) \ge n+1$, we have

$$T\left(r, \frac{h_j}{h_i}\right) \le \frac{q(q-2)}{2}\left(\sum_{i=1}^{p+1}\frac{1}{k_i}T_{f^1}(r) + \sum_{i=1}^{2n+2}\frac{1}{k_i}T_{f^2}(r)\right) + o(T(r)), \quad \forall\, j \in S(i),$$

and

$$T\left(r, \frac{h_j}{h_i}\right) \le 2\frac{q(q-2)}{2}\left(\sum_{i=1}^{p+1}\frac{1}{k_i}T_{f^1}(r) + \sum_{i=1}^{2n+2}\frac{1}{k_i}T_{f^2}(r)\right) + o(T(r)), \quad \forall\, j \notin S(i),$$

where $q = \binom{p+1}{n+2}$.

Without loss of generality, we prove the claim for $i = n + 2$. Indeed, suppose contrarily that $\sharp S(n+2) \le n$,we may assume that $1,\ldots,n+1 \notin S(n+2)$. Put $I_{01} = (1,\ldots,n+1)$ and suppose that $I_{01} \in \mathscr{I}_1$. Since $f^1 \times f^2$ is algebraically non-degenerate,

$$\sum_{s=0}^{n_1+1} A_{I_{s1}}(z)\frac{W^{I_{s1}}(z)}{V^{I_{s1}}(z)} \equiv 0,\ \forall z \in \mathbb{C}^m \text{ where } W^I = \prod_{i\in I}\sum_{j=0}^{n} a_{ij}(z)\omega_j \text{ and } V^I(z) = \prod_{i\in I}\sum_{j=0}^{n} a_{ij}(z)v_j.$$

We take a point z_0 which is not zero neither pole of any A_I, not pole of any a_{ij}, not in the indeterminacy loci of all a_i and such that the family of hyperplanes $\{a_i(z_0)\}_{i=1}^{p+1}$ is in general position in $\mathbb{P}^n(\mathbb{C})$. For the point $\omega = (\omega_0:\ldots:\omega_n) \in \bigcap_{i=n+3}^{2n+2} a_i(z_0)$, we have $\sum_{\substack{I_{s1}\subset\{1,\ldots,n+2\}\\ s=0}}^{n_1+1} A_{I_{s1}}(z_0)\frac{W^{I_{s1}}(z_0)}{V^{I_{s1}}(z_0)} \equiv 0.$

So there exists $1 \le s' \le n_1 + 1$ such that $I_{s'1} \subset \{1,\ldots,n+2\}, I_{s'1} \ne I_{01}$. Therefore $n+2 \in I_{s'1}$ and $\frac{h_{I_{01}}}{h_{I_{s'1}}} = \frac{h_{n+2}}{h_j}$, for some $j \in \{1,\ldots,n+1\}$. This contradicts the supposition that $1,\ldots,n+1 \notin S(n+2)$. Hence, we must have $\sharp S(n+2) \ge n+1$. Now for $j \in S(n+2)$, we may assume that there exist $\mathscr{I}_t$ two elements $I, I' \in \mathscr{I}_1$ satisfying

$$\frac{h_I}{h_{I'}} = \frac{h_j}{h_{n+2}}.$$

Assuming that F_1 has a reduced representation $F_1 = (uA_{I_{01}}h_{I_{01}}:\ldots:uA_{I_{n_1 1}}h_{I_{n_1 1}})$,where u is a meromorphic function. Thus, by the second main theorem, we have

$$\begin{aligned}
T\left(r,\frac{h_j}{h_{n+2}}\right) &\le T(r,F_1) \le \sum_{i=0}^{n_1+1} N^{[n_1]}_{uA_{I_{i1}}h_{I_{i1}}} + o(T(r))\\
&\le \sum_{i=0}^{n_1+1}\left(N^{[n_1]}_{A_{I_{i1}}h_{I_{i1}}}(r) + N^{[n_1]}_{1/A_{I_{i1}}h_{I_{i1}}}(r)\right) + o(T(r))\\
&\le \sum_{i=0}^{n_1+1}\left(N^{[n_1]}_{h_{I_{i1}}}(r) + N^{[n_1]}_{1/h_{I_{i1}}}(r)\right) + o(T(r))\\
&\le n_1\sum_{i=0}^{n_1+1}\left(N^{[1]}_{h_{I_{i1}}}(r) + N^{[1]}_{1/h_{I_{i1}}}(r)\right) + o(T(r))\left(r,F_1\right) \le \sum_{i=0}^{n_1+1} N^{[n_1]}_{uA_{I_{i1}}h_{I_{i1}}}(r) + o(T(r))\\
&\le n_1\sum_{I\in\mathbf{I}}\left(N^{[1]}_{h_I}(r) + N^{[1]}_{1/h_I}(r)\right) + o(T(r))\\
&\le n_1\sum_{I\in\mathbf{I}}\sum_{i\in I}\left(N^{[1]}_{h_i}(r) + N^{[1]}_{1/h_i}(r)\right) + o(T(r))\\
&= \frac{n_1 q}{2}\sum_{i=1}^{2n+2}\left(N^{[1]}_{h_i}(r) + N^{[1]}_{1/h_i}(r)\right) + o(T(r))\\
&= \frac{n_1 q}{2}\left(\sum_{i=1}^{p+1}(N^{[1]}_{h_i}(r) + N^{[1]}_{1/h_i}(r)) - \sum_{i=2n+2}^{p+1}(N^{[1]}_{h_i}(r) + N^{[1]}_{1/h_i}(r))\right) + o(T(r))
\end{aligned}$$

$$\le \frac{n_1 q}{2}\left(\sum_{i=1}^{p+1}(N^{[1]}_{(f^1,a_i),\ge k_i}(r)+N^{[1]}_{(f^2,a_i),\ge k_i}(r))-\sum_{i=2n+2}^{p+1}N^{[1]}_{(f^2,a_i),\ge k_i}(r)\right)+o(T(r))$$

$$\le \frac{q(q-2)}{2}\left(\sum_{i=1}^{p+1}\frac{1}{k_i}N_{(f^1,a_i),\ge k_i}(r)+\frac{1}{k_i}\sum_{i=1}^{2n+2}N_{(f^2,a_i),\ge k_i}(r)\right)+o(T(r))$$

$$\le \frac{q(q-2)}{2}\left(\sum_{i=1}^{p+1}\frac{1}{k_i}\mathrm{T}_{f^1}(r)+\sum_{i=1}^{2n+2}\frac{1}{k_i}\mathrm{T}_{f^2}(r)\right)+o(T(r)).$$

For $j\notin S(n+2)$, since $\sharp S(j)\ge n+2$, there exists $l\in S(j)\cap S(n+2)$. Hence, by the above proof, we have

$$T\left(r,\frac{h_j}{h_{n+2}}\right)\le 2\frac{q(q-2)}{2}\left(\sum_{i=1}^{p+1}\frac{1}{k_i}\mathrm{T}_{f^1}(r)+\sum_{i=1}^{2n+2}\frac{1}{k_i}\mathrm{T}_{f^2}(r)\right)+o(T(r)).$$

The claim is proved.

We see that there exist $b_{ij}\in\mathscr{R}\{a_i\}_{i=1}^{p+1}\,(n+2\le i\le 2n+2, 1\le j\le n+1)$ such that

$$\tilde{a}_i=\sum_{j=1}^{n+1}b_{ij}\tilde{a}_j,\ n+2\le i\le 2n+2.$$

From (3), we have

$$\psi:=(\tilde{a}_{i0}h_i-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{j0}h_j,\ldots,\tilde{a}_{in}h_i-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{jn}h_j;\ n+2\le i\le 2n+2).$$

has the rank at most.

Suppose that $rank\ \psi<n$. Then, the determinant of the square submatrix of ψ

$$\det(\tilde{a}_{i1}h_i-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{j0}h_j,\ldots,\tilde{a}_{in}h_i-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{jn}h_j;\ n+2\le i\le 2n+1)=0.$$

By Lemma 3.2, it follows that $f^1\times f^2$ is algebraically degenerate over $\mathscr{R}\{a_i\}_{i=1}^{p+1}$. This contradicts the supposition. Hence $rank\ \psi$ = n.

On the other hand, we have

$$(\tilde{a}_{i0}h_i-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{j0}h_j)f^t_0+\ldots+(\tilde{a}_{in}h_i-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{jn}h_j)f^t_n=0,\ n+2\le i\le 2n+1, t=1,2.$$

Thus

$$(\tilde{a}_{i0}\frac{h_i}{h_1}-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{j0}\frac{h_j}{h_1})\frac{f^t_0}{f^t_n}+\ldots+(\tilde{a}_{i(n-1)}\frac{h_i}{h_1}-\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{j(n-1)}\frac{h_j}{h_1})\frac{f^t_{n-1}}{f^t_n}$$

$$=-\tilde{a}_{in}\frac{h_i}{h_1}+\sum_{j=1}^{n+1}b_{ij}\tilde{a}_{jn}\frac{h_j}{h_1},\ n+2\le i\le 2n+1, t=1,2.$$

We consider the above identities as a system of n equations that its solution is $\frac{f^t_i}{f^t_n}\,(0\le i\le n-1, t=1,2)$ which has the form

$$\frac{f_i^t}{f_n^t} = \frac{P_i^t}{Q_i^t}, t = 1,2$$

where $P_i^t, Q_i^t\ (t = 1,2)$ are homogeneous polynomials in $\frac{h_i}{h_1}\ (1 \le i \le 2n+1)$ of degree n. Then by Proposition 2.1 and the above Claim, we have

$$T(r) \le \sum_{i=0}^{n-1}\left(T(\mathrm{r}, \frac{f_i^1}{f_n^1}) + T(\mathrm{r}, \frac{f_i^2}{f_n^2})\right) = \sum_{i=0}^{n-1}\left(T(r, \frac{P_i^1}{Q_i^1}) + T(\mathrm{r}, \frac{P_i^2}{Q_i^2})\right) \le 2n^2 \sum_{i=1}^{2n+1} T(r, \frac{h_i}{h_1}) + o(T(r))$$

$$\le \frac{n^2 q(q-2)}{2} \sum_{2\le i\le 2n+1, i\in S(1)} \left(\sum_{i=1}^{p+1} \frac{1}{k_i} T_{f^1}(r) + \sum_{i=1}^{2n+2} \frac{1}{k_i} T_{f^2}(r)\right)$$

$$+ \frac{2n^2 q(q-2)}{2} \sum_{2\le i\le 2n+1, i\notin S(1)} \left(\sum_{i=1}^{p+1} \frac{1}{k_i} T_{f^1}(r) + \sum_{i=1}^{2n+2} \frac{1}{k_i} T_{f^2}(r)\right)$$

$$\le \frac{3n^2 q(q-2)}{2} \sum_{i=1}^{p+1} \frac{1}{k_i} T_{f^1}(r) + \frac{3n^2 q(q-2)}{2} \sum_{i=1}^{2n+2} \frac{1}{k_i} T_{f^2}(r)$$

where the last inequality comes from the fact that there are at most n indices $i \notin S(1)$.

Thus

$$\left(1 - \frac{3n^2 q(q-2)}{2} \sum_{i=1}^{p+1} \frac{1}{k_i}\right) T_{f^1}(r) + \left(1 - \frac{3n^2 q(q-2)}{2} \sum_{i=1}^{2n+2} \frac{1}{k_i}\right) T_{f^2}(r) \le \mathrm{o(T(r))}.$$

This implies that

$$A_1 T_{f^1}(r) + A_2 T_{f^2}(r) \le o(T(r)).$$

Letting $r \to +\infty$., we have a contradiction with (1) and (2). This is a contradiction. Hence, $f^1 \times f^2$ is algebraically degenerate. The theorem is proved.

4. Conclusions

In this article, we proved an algebraic dependence theorem for meromorphic mappings into a projective space. This is an extension of S. D. Quang's result [12] in the case where these mappings not only have the same inverse images counted different multiplicities but also consider the weaker condition about this multiplicities among the given moving hyperplanes.

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