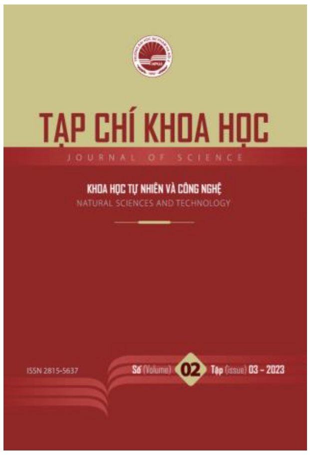
HPU2. Nat. Sci. Tech. Vol 02, issue 03 (2023), 03-12

HPU2 Journal of Sciences:
Natural Sciences and Technology

journal homepage: https://sj.hpu2.edu.vn




***Article type: Research article***

# The Koszulness of numerical semigroup rings of minimal multiplicity

Van-Kien Do[a,*], Hong-Ngoc Nguyen[a],

[a] *Hanoi Pedagogical University 2, Vinh Phuc, Vietnam*

**Abstract**

Let $R=k[\![H]\!]$ be a numerical semigroup ring over a field $k$ and $\mathrm{gr}_{\mathfrak{m}}(R)$ is the associated graded ring of $R$. In this paper, we show that $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring, provided $H$ has minimal multiplicity. As a consequence, we conclude that the numerical semigroup ring $R=k[\![H]\!]$ of minimal multiplicity is a Koszul ring, i.e., the residue field $k$ has a $\mathrm{gr}_{\mathfrak{m}}(R)-$linear free resolution.

***Keywords:*** Cohen-Macaulay ring; koszul ring; numerical semigroup ring; associated graded ring

# Tính Koszul của vành nửa nhóm số có bội tối tiểu

Đỗ Văn Kiên[a]*, Nguyễn Hồng Ngọc[a]

[a]*Trường Đại học Sư phạm Hà Nội 2, Vĩnh Phúc, Việt Nam*

**Tóm tắt**

Cho $R=k[\![H]\!]$ là một vành nửa nhóm số trên một trường $k$ và $\mathrm{gr}_{\mathfrak{m}}(R)$ là vành phân bậc liên kết của *R*. Trong bài báo này, chúng tôi chỉ ra rằng vành phân bậc $\mathrm{gr}_{\mathfrak{m}}(R)$ là một vành Cohen-Macaulay nếu

[*] Corresponding author, E-mail: dovankien@hpu2.edu.vn

https://doi.org/10.56764/hpu2.jos.2023.2.3.3-12

Received date: 27-9-2023 ; Revised date: 01-12-2023 ; Accepted date: 08-12-2023

nửa nhóm *H* có bội tối tiểu. Chúng tôi áp dụng kết quả này để chỉ ra rằng vành nửa nhóm số có bội tối tiểu $R = k[\![H]\!]$ là một vành Koszul, tức là trường thặng dư *k* có một giải tự do tuyến tính trên $\mathrm{gr}_{\mathfrak{m}}(R)$.
***Từ khóa:*** Vành Cohen-Macaulay; Vành Koszul; Vành nửa nhóm số; Vành phân bậc liên kết.

## 1. Introduction

Let *k* be a field and $(R,\mathfrak{m})$ be a standard graded *k*-algebra with the graded maximal ideal $\mathfrak{m}$. We say that $R$ is *Koszul* (or a *Koszul algebra*) if the residue field $k = R/\mathfrak{m}$ has a linear free resolution over $R$ of the form

$$(\sharp) \qquad \cdots \to R(-i)^{\beta_i} \to \cdots \to R(-2)^{\beta_2} \to R(-1)^{\beta_1} \to R \to k\,.$$

This means that the matrices describing the differentials of $k$ in $(\sharp)$ have non-zero entries only linear forms. Koszul algebras were originally introduced by Priddy ([13]) in his study of the homological properties of graded algebras. We refer to the survey articles/books ([5, 7, 12]) for more details. There are important conections between the Koszulness and the structure of non-commutative algebra $\mathrm{Ext}_R^*(k,k)$, i.e., the Yoneda-Hopf algebra of *k*. Among other things, Koszul algebras are also important because they give an interesting class of quadratic algebras with rational Poincaré series.

Note that if $R$ is a standard graded $k$-algebra, then there is a presentation $R \cong k[X_1,...,X_n]/I$, where $k[X_1,...,X_n]$ is a polynomial ring. We call $I$ the defining ideal of $R$. Then the condition that $k$ has a linear resolution over $R$ is equivalent to the fact that the Betti graded numbers $\beta_{ij}^R(k) = 0$ for all $j \neq i$. In particular, $\beta_{2j}^R(k) = 0$ for all $j \neq 2$. This implies that whenever $R$ is Koszul then $I$ is quadratics, that is generated by homogeneous polynomials of degree 2. The converse does not hold in general. For instance, the ring $R = k[X,Y,Z,U]/(X^2,Y^2,Z^2,U^2,XY+XZ+XU)$ has $\beta_{34}^R(k) = 2 \neq 0$ which implies that $R$ is not Koszul, while $I = (X^2,Y^2,Z^2,U^2,XY+XZ+XU)$ is quadratic. Nevertheless, it is also well-known that (see [12, Theorem 34.12]) if $I$ has a quadratic Gröbner basis then $R$ is Koszul.

In this survey we investigate the Koszul property of the associated graded ring of a numerical semigroup ring. The paper is organized as follows. In Section 2, we will begin with some preliminaries on numerical semigroups of minimal multiplicity that we need in the paper. Section 3 consists of a survey on Koszul filtration, an ffective tool to prove an algebra to be Koszul. In the last section, we give the main results of the paper. More precisely, we show that the associated graded rings of numerical semigroup rings of minimal multiplicity is Cohen-Macaulay. As an application, we prove that numerical semigroup rings of minimal multiplicity is Koszul. We also provide a few examples to illustrate this result.

## 2. Preliminaries

### 2.1. Numerical semigroups of minimal multiplicity

Let $a_1, a_2, ..., a_n$ be a sequence of positive integers such that $\gcd(a_1, a_2, ..., a_n) = 1$. Let

$$H=\langle a_1,a_2,\ldots,a_n\rangle=\left\{\sum_{i=1}^{n}c_ia_i \,\middle|\, 0\le c_i\in\mathbb{Z} \text{ for all } 1\le i\le n\right\}.$$

Then $H$ is a submonoid of the additive monoid $\mathbb{N}$ and $H$ has a finite complement in $\mathbb{N}$. We call $H$ the *numerical semigroup* generated by $a_1,a_2,\ldots,a_n$. The system $\{a_1,a_2,...,a_n\}$ is said to be minimal if for every $1\le i\le n, a_i$ can not be written as a combination of $\{a_1,a_2,...,a_n\}\setminus\{a_i\}$ with integer coefficients. In what follows, let us denote $H$ a numerical semigroup minimally generated by $n$ elements. There are a few invariants of $H$ as follows.

**Definition 2.1.**

(1) The *embedding dimension* of $H$ denoted by edim($H$) is the cardinality of the minimal set of generators of $H$, i.e., edim($H$) = $n$.

(2) The *multiplicity* of $H$, denote by $e(H)$, is defined to be the smallest non-zero element of $H$, i.e., $e(H)=a_1$.

(3) The *Frobenius number* of $H$, denoted by $F(S)$ to be $\max(\mathbb{Z}\setminus S)$.

(4) The set of *gaps* of $H$, denoted by $G(S)$ to be $\mathbb{N}\setminus H$.

(5) The *genus* of $H$, denoted by $g(S)$ to be the cardinality of $G(S)$.

There is an interesting relation between the embedding dimension and the multiplicity that $\operatorname{edim}(H)\le e(H)$ (see [4, Proposition 2.10]). The semigroup $H$ that attains this bound is called the *minimal multiplicity* or the *maximal embedding dimension*.

Let $a\in H$ be a nonzero element. The *Apéry* set of $H$ with respect to $a$ is defined by $\operatorname{Ap}(H,a)$ $=\{h\in H \mid h-a\notin H\}$. We summarize several characterizations of the numerical semigroup of minimal multiplicity in the following proposition.

**Proposition 2.2** (see [4]). *The following conditions are equivalent*.

*1) $H$ has minimal multiplicity.*

*2) For all $x,y\in H$ such that $x\ge y\ge e(H)$ then $x+y-e(H)\in H\setminus\{0\}$.*

*3)* $\operatorname{Ap}(H,a_1)=\{0,a_2,...,a_n\}$.

*4)* $g(S)=\dfrac{1}{a_1}\sum_{i=2}^{n}a_i-\dfrac{a_1-1}{2}$.

## 2.2. Numerical semigroup rings and their associated graded rings

Let $H=\langle a_1,a_2,...,a_n\rangle$ be a numerical semigroup of embedding dimension $n$. Let $k$ be a field and $R=k\ H\ =k[\![t^{a_1},...,t^{a_n}]\!]$ is a subring of the formal power series ring $k\ t$. Then $R$ is a Noetherian local domain of dimension 1 with maximal ideal $\mathfrak{m}=(t^{a_1},...,t^{a_n})$. We call $R$ the *semigroup ring* associated to $H$. It is easy to see that $e(H)=e(R)$, $\operatorname{edim}(H)=\operatorname{edim}(R)$, whence $H$ has minimal multiplicity if and only if $R$ has minimal multiplicity. A classical result connecting properties of the semigroup to properties of the ring due to Kunz [10] states that $R$ is

Gorenstein if and only if $H$ is symmetric. Others have linked properties of the semigroup to properties of the associated graded ring of R, $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$ which is a standard graded ring of dimension 1. Let $x_i$ denote the image of $t^{a_i}$ in $\mathfrak{m}/\mathfrak{m}^2$ for all $i$. Then $\mathrm{gr}_{\mathfrak{m}}(R) \cong k[x_1, ..., x_n]$ where a "monomial" $x_1^{c_1}...x_n^{c_n}$ is non-zero if and only if $\sum_{i=1}^{n} c_i = \max\deg\left(\sum_{i=1}^{n} c_i a_i\right)$. If $x_1^{c_1}...x_n^{c_n} \neq 0$ then $x_1^{c_1}...x_n^{c_n} = x_1^{d_1}...x_n^{d_n}$ if and only if $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} d_i$ and $\sum_{i=1}^{n} c_i a_i = \sum_{i=1}^{n} d_i a_i$. Here, for $h \in H$, $\max\deg(h) = \max\left\{\sum_{i=1}^{n} c_i \,\middle|\, h = \sum_{i=1}^{n} c_i a_i\right\}$.

García ([8]) showed the following interesting result.

**Theorem 2.3** ([8], Theorem 7). *Let* $R = k[\![t^{a_1}, ..., t^{a_n}]\!]$ *be a numerical semigroup ring. Then* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay if and only if* $x_1$ *is a non-zero divisor in* $\mathrm{gr}_{\mathfrak{m}}(R)$.

Several authors tried to find some classes of semigroup rings such that their associated graded rings are Cohen–Macaulay or at least have non-decreasing Hilbert function. D'Anna, Micale and Sammartano in [17, 18] characterized when $\mathrm{gr}_{\mathfrak{m}}(R)$ is Buchsbaum/complete intersection. For the case where the embedding dimension is 3, Robbiano and Valla [14] gave necessary and sufficient conditions on the generators of the defining ideal for the associated graded ring to be a complete intersection and for it to be Cohen-Macaulay.

Note that $R$ and its associated graded ring have the same residue field. We say that $R$ is *Koszul* if $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Koszul algebra, that is, the residue field has a linear free resolution over $\mathrm{gr}_{\mathfrak{m}}(R)$.

## 3. Koszul filtrations

A useful tool to attack the Koszulness is through Koszul filtrations. Concretely, one possible way to affirm the Koszulness of an algebra is to show that it admits a Koszul filtration. We first recall the notion of Koszul filtrations introduced by Conca, Trung and Valla in [6].

**Definition 3.1.** Let $G$ be a standard graded ring. A family $\mathcal{F}$ of ideals of $G$ is said to be a *Koszul filtration* of $G$ if:

1) Every ideal $I \in \mathcal{F}$ is generated by linear forms.

2) The ideal $(0)$ and the maximal graded ideal $\mathfrak{m}$ of $G$ belong to $\mathcal{F}$.

3) For every $I \in \mathcal{F}$ different from $(0)$, there exists $J \in \mathbb{F}$ such that $J \subseteq I$, $I/J$ is cyclic and $J : I \in \mathcal{F}$.

In [6], it is proved that all the ideals belonging to such a filtration have a linear free resolution over $G$ and in particular, since the graded maximal ideal $\mathfrak{m} \in \mathcal{F}$, $G$ will be a Koszul algebra. Using the Koszul filtration we have the following interesting fact.

**Lemma 3.2.** *Let* $I$ *be a monomial ideal generated by quadrics in a polynomial ring* $S = k[X_1, X_2, ..., X_n]$. *Then the ring* $G = S/I$ *is Koszul.*

*Proof*. Denote $x_i$ is the image of $X_i$ in $G$ for all $1 \le i \le n$. Let $\mathcal{F}$ is the set of all ideals in $G$ generated by variables and let $M_1, M_2, ..., M_s$ be a minimal system of monomial generators for $I$. Note that for any ideal $J \subseteq R$ generated by variables and any $x_i \notin J$, the colon ideal $J : x_i$ is equal to $J + (x_j$ such that $X_j$ divides some $M_\ell$). The ideal $J : x_i$ is therefore generated by variables and it belongs to $\mathcal{F}$. The family $\mathcal{F}$ is a Koszul filtration because any ideal in $\mathcal{F}$ can be filtered simply by dropping one variable by its minimal generators. By the maximal ideal $x_1, x_2, ..., x_n \in \mathcal{F}$ has a linear free resolution and consequently $G$ is Koszul.

□

## 4. Main results

Throughout this section, let $H = \langle a_1, a_2, ..., a_n \rangle$ be the numerical semigroup with $a_1 < a_2 < ... < a_n$. Let $k$ be an infinite field. We set $R = k\ H\ = k[\![t^{a_1}, t^{a_2}, ..., t^{a_n}]\!]$ is the numerical semigroup ring associated to $H$ over $k$, where $t$ is an indeterminate. It is clear that $R$ is a Cohen-Macaulay local ring of dimension 1 with the maximal ideal $\mathfrak{m} = \left(t^{a_1}, t^{a_2}, ..., t^{a_n}\right)$. Let $T = k\ X_1, X_2, ..., X_n$ be the formal power series and $S = k\left[X_1, X_2, ..., X_n\right]$ the polynomial ring over $k$. Let $\varphi : T \to R$ denote the homomorphism of $k$-algebras defined by $\varphi\left(X_i\right) = t^{a_i}$ for all $1 \le i \le n$. Let $I = \operatorname{Ker} \varphi$ be the defining ideal of $R$. It is known that $I \subseteq \left(X_1, X_2, ..., X_n\right)^2$ and $I$ is generated by the binomials $\prod_{i=1}^{n} X_i^{\alpha_i} - \prod_{i=1}^{n} X_i^{\beta_i}$ with $\alpha_i, \beta_i \ge 0$ and $\sum_{i=1}^{n} c_i \alpha_i = \sum_{i=1}^{n} d_i \beta_i$. Let $G = \operatorname{gr}_{\mathfrak{m}}\left(R\right) = \bigoplus_{i \ge 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$ the associated graded ring of $R$ with respect to the maximal ideal $m$. Then $G$ is a standard graded ring with the $i$-th graded component $G_i = \mathfrak{m}^i / \mathfrak{m}^{i+1}$. Let $\mathfrak{n} := \left(X_1, X_2, ..., X_n\right)$ denote the maximal ideal of $T$. Then $\varphi$ induces an epimorphism $\overline{\varphi} : \operatorname{gr}_{\mathfrak{n}}\left(T\right) \to \operatorname{gr}_{\mathfrak{m}}\left(R\right)$. But $\operatorname{gr}_{\mathfrak{n}}\left(T\right) = \bigoplus_{i \ge 0} \mathfrak{n}^i / \mathfrak{n}^{i+1} \cong k\left[X_1, X_2, ..., X_n\right] = S$. Hence there is an epimorphism from $S$ on $G$. Denote $I^* = \left(f^* \mid f \in I \setminus \{0\}\right)$ the ideal generated by initial forms of $I$, where $f^*$ is the homogeneous component of $f$ of least degree. Then $G \cong S / I^* = k\left[X_1, X_2, ..., X_n\right] / I^*$. So we can identity $G_i = \left(S_i + I^*\right) / I^*$ for all $i \ge 0$ so that $G = S / I^*$. The maximal graded ideal of $G$ is $\mathcal{M} = \mathfrak{n} / I^*$. We call $I^*$ is the defining ideal of $G$. We begin with the following which implies that $G$ is Cohen-Macaulay, provided $R$ has minimal multiplicity.

Note that this result is also found in [16, Theorem 2], but we provide a more elementary proof.

**Proposition 4.1.** *Suppose that $R$ has minimal multiplicity. Then its associate graded ring $G$ is Cohen-Macaulay.*

*Proof*. Firstly, we show that $\mathfrak{m}^2 = t^{a_1}\mathfrak{m}$. The inclusion $\mathfrak{m}^2 \supseteq t^{a_1}\mathfrak{m}$ is clear. For converse, take any $t^{a_i + a_j} \in \mathfrak{m}^2, 1 \le i, j \le n$. Clearly, $t^{a_i + a_j} \in t^{a_1}\mathfrak{m}$ once either $a_i = a_1$ or $a_j = a_1$. If $a_i \ne a_1$ and $a_j \ne a_1$ then by Proposition 2.2(2) we have $a_i + a_j - a_1 \in H \setminus \{0\}$. This yields that $t^{a_i+a_j-a_1} \in \mathfrak{m}$, whence $t^{a_i+a_j} \in t^{a_1}\mathfrak{m}$. Thus $\mathfrak{m}^2 = t^{a_1}\mathfrak{m}$ and hence $\mathfrak{m}^{s+1} = t^{a_1}\mathfrak{m}^s$ for all $s \geq 1$.

Denote $x_i$ the images of $t^{a_i}$ in $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, for all $i \geq 0$. One has $x_1 = t^{a_1} + \mathfrak{m}^2$. Suppose that $x_1.(g + \mathfrak{m}^{j+1}) = 0$ in $G$ with $g \in \mathfrak{m}^j$ for some $j \geq 0$. Then $(t^{a_1} + \mathfrak{m}^2)(g + \mathfrak{m}^{j+1}) = 0$ in $G$. This is equivalent to $t^{a_1} g \in \mathfrak{m}^{j+2}$. Because $\mathfrak{m}^{j+2} = t^{a_1}\mathfrak{m}^{j+1}$, $g \in \mathfrak{m}^{j+1}$. This follows that $g + \mathfrak{m}^{j+1} = 0$ in $G$. So, $x_1$ is non-zero divisor in $G$. Hence thanks to Theorem 2.3, $G$ is a Cohen-Macaulay ring.

□

**Lemma 4.2.** *There is a homogeneous element $x \in G_1$ such that $\left(0 \underset{G}{:} x\right)$ has the finite length, i.e., $\ell_G(0 : x) < \infty$.*

*Proof*. We consider an ascending sequence of ideals in $G$ as follows

$$0 \underset{G}{:} \mathcal{M} \subseteq 0 \underset{G}{:} \mathcal{M}^2 \subseteq 0 \underset{G}{:} \mathcal{M}^3 \cdots.$$

Since $G$ is Notherian, there exists $n_0 \geq 0$ such that $0 \underset{G}{:} \mathcal{M}^{n_0} = 0 \underset{G}{:} \mathcal{M}^{n_0+1} = \cdots$. This implies that $L = 0 \underset{G}{:} \mathcal{M}^{n_0}$, where $L = \bigcup_{i\geq 0}\left(0 \underset{G}{:} \mathcal{M}^i\right)$. Hence $\mathcal{M}^{n_0} L = 0$. This yields $\ell_G(L) < \infty$. We set Ass $\left(G/L\right) = \{P_1, \ldots, P_r\}$ the set of associated prime ideals of $G/L$. If $\mathcal{M} \in G/L$ then, by the definition of associated prime ideals, there is an element $\eta + L \in G/L$, $\eta \notin L$ such that $\mathcal{M} = 0 : (\eta + L)$. Hence $\eta\mathcal{M} \subseteq L$ which implies $\eta\mathcal{M}^{n_0+1} = 0$ (because $\mathcal{M}^{n_0} L = 0$). It follows that $\eta \in 0 : \mathcal{M}^{n_0+1} = L$ which is a contradiction. This says that $\mathcal{M} \neq P_i$ for all $i = 1, \ldots, r$. We see that $P_i \cap G_1$ is a proper *k*-vector space of $G_1$. Because if otherwise, $P_i \cap G_1 = G_1$ for some $1 \leq i \leq r$. Then $P_i \supseteq G_1$. In particular, $(X_j + I^*)/I^* \in P_i$ for all $1 \leq j \leq n$, whence $P_i = \mathcal{M}$. This is impossible. So, $G_1 \setminus (P_i \cap G_1) \neq \varnothing$ for all $1 \leq i \leq r$. On the other hand, by the assumption $|k| = \infty$, $G_1 \setminus \bigcup_{i=1}^{r}(P_i \cap G_1) \neq \varnothing$. This implies that there is an element $x \in G_1$, $x \notin P_i$ for all $1 \leq i \leq r$. Since $x \notin P$ for all $P \in \text{Ass}\left(G/L\right)$, $x$ is a non-zero divisor on $G/L$. Now we take any $f \in 0 \underset{G}{:} x$. Then $x(f + L) = xf + L = L$. Because *x* is a non-zero divisor on $G/L$, we get $f + L = L$ which implies $f \in L$. So, $0 \underset{G}{:} x \subseteq L$. But $\ell_G(L) < \infty$, we get that $\ell_G\left(0 \underset{G}{:} x\right) < \infty$ as desired.

□

With the element $x$ as in Lemma 4.2, we set $\overline{G} = G/xG$.

**Proposition 4.3.** *Suppose $R$ has minimal multiplicity. Then $x$ is regular on $G$ and* $e(G)=e(\overline{G})$.

*Proof*. We consider two the following exact sequences

$$0\to \frac{G}{0\underset{G}{:}x}(-1)\xrightarrow{x} G\to \overline{G}\to 0,$$

$$0\to 0\underset{G}{:}x\to G\to G/0\underset{G}{:}x\to 0.$$

Denote $H_G(t)$ the Hilbert series of $G$. Since $\dim G=\dim R=1$, $H_G(t)=\dfrac{q(t)}{1-t}$ for some $q(t)\in\mathbb{Q}[t]$ (see [12, Theorem 16.7]), with $q(1)=e(G)$. Using [12, Theorem 16.1], we get

$$\begin{aligned} H_{\overline{G}}(t) &= H_G(t)-H_{\frac{G}{0\underset{G}{:}x}(-1)}(t)\\ &= H_G(t)-tH_{\frac{G}{0\underset{G}{:}x}}(t)\\ &= H_G(t)-t\left(H_H(t)-H_{0\underset{G}{:}x}(t)\right)\\ &=(1-t)H_G(t)+tH_{0\underset{G}{:}x}(t)\\ &=q(t)+tH_{0\underset{G}{:}x}(t) \qquad (*) \end{aligned}$$

Since $R$ has minimal multiplicity, by Proposition 4.1, $G$ is a one dimensional Cohen-Macaulay ring. This implies $H^0_{\mathcal{M}}(G)=0$, i.e., $\bigcup_{i\geq 0}\left(0\underset{G}{:}\mathcal{M}^i\right)=0$, or equivalently, $L=0$. It implies that $0\underset{G}{:}x=0$ because $0\underset{G}{:}x\subseteq L$. This concludes that $x$ is regular on $G$ and $\overline{G}$ is Artinian. Hence, by the equality $(*)$, we get $e(\overline{G})=H_{\overline{G}}(1)=q(1)=e(G)$.

□

We now show that if $R$ has minimal multiplicity then it is Koszul.

**Theorem 4.4.** *Suppose that $H$ has minimal multiplicity. Then the semigroup ring $R=\ H$ is Koszul.*

*Proof*. With the element $x$ as in Lemma 4.2, we set $\overline{G}=G/xG$. By the definition, we need to show the associated graded ring $G$ of $R$ is a Koszul algebra. Note that $G$ and $\overline{G}$ or both are Koszul or both are not Koszul (see [12, Exercise 34.14]). So, it is sufficient to show that $\overline{G}$ is Koszul. Since $x$ is a homogeneous element of degree 1 in $G=S/I^*$, $x$ has the form $x=f+I^*$, where $f\in S_1$. Without loss of generality, we may assume $f=X_1+X_2+...+X_s$ for some $1\leq s\leq n$. Then

$$\overline{G} = G/xG = \frac{k[X_1,...,X_n]/I^*}{(X_1+...+X_s+I^*)} \cong \frac{k[X_1,...,X_n]}{(X_1+...+X_s)+I^*} .$$

By dividing polynomials in $I^*$ by $X_1$ we can write $(X_1+...+X_s)+I^* = (X_1+...+X_s)+J$ with $J \subseteq k[X_2,...,X_s]$. Then one has

$$\begin{aligned}\overline{G} &\cong \frac{k[X_1,...,X_n]}{(X_1+...+X_s)+J} \\ &\cong \frac{k[X_1,...,X_n]/(X_1+\cdots+X_s)}{((X_1+\cdots+X_s)+J)/(X_1+\cdots+X_s)} \\ &\cong k[X_2,...,X_n]/J .\end{aligned}$$

The last equality follows from $k[X_1,...,X_n]/(X_1+\cdots+X_s) \cong k[X_2,...,X_n]$ and $(X_1+...+X_s)\cap J = 0$.

Thus it is enough to prove that $k[X_2,...,X_n]/J$ is Koszul. Indeed, we have

$$\begin{aligned}e(\overline{G}) = \ell(\overline{G}) &= \dim_k\left(k[X_2,...,X_n]/J\right) \\ &= \dim_k k[X_2,...,X_n]/(X_2,...,X_n)^2 - \dim_k\left((X_2,...,X_n)^2/J\right) \\ &= n - \dim_k\left((X_2,...,X_n)^2/J\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}e(G) \geq \operatorname{edim}(G) &= \dim_k \frac{\mathfrak{n}/I^*}{(\mathfrak{n}/I^*)^2} = \dim_k\left(\mathfrak{n}/\mathfrak{n}^2\right) \text{ (because } I^* \subseteq \mathfrak{n}^2\text{)} \\ &= \dim_k k[X_1,...,X_n]/(X_1,...,X_n)^2 - \dim_k k[X_1,...,X_n]/(X_1,...,X_n) \\ &= n+1-n = n.\end{aligned}$$

Since $H$ has minimal multiplicity, so is $R$. By Proposition 4.3, we have $e(\overline{G}) = n - \dim_k (X_1,...,X_n)^2/J = e(G) \geq n$ which implies that $(X_1,...,X_n)^2/J = 0$. Hence $J = (X_1,...,X_n)^2$. Finally, it is easy to see that $k[X_2,...,X_n]/(X_1,...,X_n)^2$ is Koszul because it possesses the Koszul filtration $\mathcal{F} = \{0,(X_2),(X_2,X_3),...,(X_2,X_3,...,X_n)\}$ in the sense of Definition 3.1 (or also by Lemma 3.2). Thus $\overline{G}$ and hence $G$ is Koszul.

□

We close this article with a few the following examples. Here we use Macaulay2 (see [11]) to compute Betti diagrams.

**Example 4.5.** Let $H = \langle 4,11,14,17 \rangle$ and $R = k[\![H]\!]$. Let $G$ is the associated graded ring of $R$. Then since $R$ has minimal multiplicity, $G$ is Koszul. The fact that in this example the defining ideal of $G$ is $I^* = (u^2, zu, yu, z^2, yz, y^2)$, that is, $G \cong k[x, y, z, u]/(u^2, zu, yu, z^2, yz, y^2)$. We have the Betti diagram of $k$ over $G$:

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 12 | 36 | 108 | 324 |

This follows that $\beta_{ij}^G(k) = 0$ for all $i \neq j$.

Let us give the following two examples. In which the semigroup ring $R$ has no minimal multiplicity while one of them is Koszul and the other one is not Koszul.

**Example 4.6.** Let $H = \langle 8,10,11,12 \rangle$ and $R = k[\![H]\!]$. Let $G$ is the associated graded ring of $R$. In this example the defining ideal of $G$ is $I^* = (u^2, z^2 - yu, y^2 - xu)$, that is, $G \cong k[x, y, z, u] \big/ (u^2, z^2 - yu, y^2 - xu)$. The Betti diagram of $k$ over $G$ is given by

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 9 | 16 | 25 | 36 |

This yields that $\beta_{ij}^G(k) = 0$ for all $i \neq j$ which implies that $G$ also is Koszul. Note that because $e(R) = 8 > \text{edim}(R) = 4$, $R$ has no minimal multiplicity.

**Example 4.7.** Let $H = \langle 12,14,15,16,18,19 \rangle$ and $R = k[\![H]\!]$. Let $G$ is the associated graded ring of $R$. Then the defining ideal of $G$ is

$$I^* = (w^2, v^2, uv - xw, zv - yw, u^2 - yv, zu - xw, yu - xv, z^2 - xv, y^2 - xu).$$

In this example, the Betti diagram of $k$ over $G$ is given by

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 6 | 24 | 84 | 276 | 877 |
| 1 | 0 | 0 | 0 | 0 | 1 | 14 |

We see that $\beta_{45}^G(k) = 1 \neq 0$ which implies that $G$ is not Koszul. Note that in this case, $R$ has no minimal multiplicity because $e(R) = 12 > \text{edim}(R) = 4$.

## 5. Conclusions

The article focuses on the Koszul property of semigroup rings. We provide a brief proof to show that the associated graded rings of numerical semigroup rings of minimal multiplicity are Cohen-

Macaulay (Proposition 4.1). Based on the Cohen-Macaulayness we show that numerical semigroup rings of minimal multiplicity are Koszul (Theorem 4.4).

**Acknowledgments**

This work is funded by the Foundation for Science and Technology Development, Hanoi Pedagogical University 2 via the grant number HPU2.2023-UT-09.

**References**

[1] S. S. Abhyankar, "Local rings of high embedding dimension", *Amer. J. Math.*, 89, pp.1073-1077, 1967, doi: 10.2307/2373418

[2] L. L. Avramov and D. Eisenbud, "Regularity of modules over a Koszul algebra", *J. Algebra*, 153, pp.85-90, 1992, doi: 10.1016/0021-8693(92)90149-G

[3] L. L. Avramov and I. Peeva, "Finite regularity and Koszul algebras", *Amer. J. Math.*, 123, pp.275-281, 2001, doi: 10.1353/ajm.2001.0008

[4] J. C. Rosales and García-Sánchez, "Numerical Semigroups", vol. 20, *Springer Science & Business Media*, 2009, doi: 10.1007/978-1-4419-0160-6

[5] A. Conca, E. De Negri, and M. E. Rossi, "Koszul Algebras and Regularity", *Comm. Algebra*, pp.285-315, Springer, New York, 2013, doi: 10.1007/978-1-4614-5292-8_8

[6] A. Conca, N. V. Trung, G. Valla, "Koszul property for points in projective spaces", *Math. Scand.*, 89, pp.201-216, 1998, doi: 10.7146/math.scand.a-14338

[7] R. Fröberg, "Koszul algebras in Advances in Commutative Ring Theory (Fez, 1997)", *Lecture Notes Pure Appl. Math.*, 205, Dekker, New York, pp.337-350, 1999, doi: 10.1201/9781003419815-21

[8] A. Garcia, "Cohen-Macaulayness of the associated graded of a semigroup rings", *Comm. Algebra*, vol. 10, no. 4, pp.393-415, 1982, doi: 10.1080/00927878208822724

[9] J. Herzog and D. I. Stamate, "Quadratic numerical semigroups and the Koszul property", *Kyoto J. Math.*, vol. 57, no. 3, pp.585-612, 2017, doi: 10.1215/21562261-2017-0007

[10] E. Kunz, "The value-semigroup of a one-dimensional Gorenstein ring", *Proc. Amer. Math. Soc.*, 25, pp.748-751, 1970, doi: 10.1090/S0002-9939-1970-0265353-7

[11] Macaulay2, "a software system for research in algebraic geometry". Available at http://www.math.uiuc.edu/Macaulay2/

[12] I. Peeva, "Graded syzygies", *Springer-Verlag London*, 2011, doi: 10.1007/978-0-85729-177-6

[13] S. B. Priddy, "Koszul resolutions", *Trans. Amer. Math. Soc.*, 152, pp.39-60, 1970, doi: 10.1090/S0002-9947-1970-0265437-8

[14] L. Robbiano and G. Valla, "On the equations defining tangent cones", *Math. Proc. Cambridge Philos. Soc.*, vol. 88, no 2, pp.28l-297, 1980, doi: 10.1017/S0305004100057583

[15] V. A. Sapko, "Associated graded rings of numerical semigroup rings", *Comm. Algebra*, vol. 29, no. 10, pp.4759-4773, 2001, doi: 10.1081/AGB-100106784

[16] J. D. Sally, "On the associated graded ring of a local Cohen-Macaulay ring", *J. Math. Kyoto Univ.*, vol. 17, pp.19-21, 1977, doi: 10.1215/kjm/1250522807

[17] M. D'Anna, V. Micale, A. Sammartano, "On the associated graded ring of a semigroup ring", *J. Commut. Algebra*, vol. 3, no. 2, pp.147-168, 2011, doi: 10.1216/JCA-2011-3-2-147

[18] M. D'Anna, V. Micale, A. Sammartano, "When the associated graded ring of a semigroup ring is complete intersection", *J. Pure Appl. Algebra*, vol. 217, no. 6, pp.1007-1017, 2013, doi: 10.1016/j.jpaa.2012.09.024