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# Solvability Analysis of high-order Linear Differential-Algebraic Equations with time-varying coefficients

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## Abstract

In this paper, we study the solvability analysis of arbitrarily high-order linear differential-algebraic equations (DAEs) with time-varying coefficients, using the algebraic-behavior approach. We propose a concept of strangeness-index and construct condensed forms for high-order linear DAEs. We also discuss other structural properties like the existence and uniqueness of a solution, consistency and smoothness requirements for an initial vector and for an inhomogeneity. Our work extends the algebraic approach for DAEs and combines this approach with the behavior approach to establish a reformulation algorithm that reveals an underlying ordinary differential equation (ODE) and all hidden constraints in the DAE. This direct treatment of the system addresses the limitations of the classical approach to transforming the system into a first-order DAE. We illustrate our theoretical results with applications in mechanical systems and electrical circuits. This comprehensive study into general high-order systems is a natural progression from the extensive study of first and second-order DAEs.

***Keywords:*** Differential-algebraic equation, strangeness-index, regularization, index reduction

## 1. Introduction

In this paper, we delve into the solvability analysis of high-order linear differential-algebraic equations (DAEs) represented by the general form:

$$A_k(t)x^{(k)}(t) + \cdots + A_1(t)\dot{x}(t) + A_0(t)\mathrm{x}(t) = \mathrm{f}(t), \qquad (1)$$

on the time interval $t \in \mathbb{I} = [t_0, t_f) \subset \mathbb{R}$, where $A_i \in C(\mathbb{I}, C^{\ell,n})$, $i = 1, \dots, k$, and $f: \mathbb{I} \to \mathbb{C}^{\ell}$. To ensure a unique solution for this equation, an initial function is required:

$$x^{(k-1)}(t_0) = x_0^{(k-1)}, \dots, \dot{x}(t_0) = \dot{x}_0, x(t_0) = x_0. \qquad (2)$$

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While first-order DAEs ($k = 1$) have been extensively studied over the past three decades with well-established theoretical and numerical results [1]–[3], second-order DAEs ($k = 2$) have found applications in diverse fields such as constrained mechanical systems [4], electrical and electro-mechanical systems [5], [6], heterogeneous systems [7], and traveling waves [8], [9], etc. Recently, the analysis and control of second-order discrete-time system has been considered in [10], [11]. Consequently, a comprehensive investigation into the general high-order system (1) is a natural progression.

Despite some exploration of specific cases ($k = 1,2$), limited results exist for the general case, and the classical approach of transforming (1) into a first-order DAE through variable introduction has shown several critical drawbacks [12]–[16]. These include excessive smoothness requirements on the inhomogeneity or numerical method failures in a linearized system. Consequently, a direct treatment of system (1) becomes imperative, forming the central objective of this paper. We adopt a unique perspective by extending the algebraic approach for DAEs proposed in previous works [3], [17] and combining it with the behavior approach [18]. The goal is to establish a reformulation algorithm that unveils an underlying ordinary differential equation (ODE) and all hidden constraints in the DAE (1).

The paper is organized as follows. Section 2 introduces necessary notations and auxiliary lemmas. In Section 3, we extend the concept of the strangeness-index, initially designed for first-order systems, to the high-order system (1). Additionally, we devise a reformulation algorithm to transform (1) into its strangeness-free form. This section also addresses the solvability analysis of the initial value problem consisting of (1)–(2). Finally, Section 4 illustrates our results through examples in the fields of mechanics and electrical circuits.

## 2. Preliminaries

In this paper we use the following solution concept for (1).

**Definition 1**. A function $x: \mathbb{I} \to \mathbb{C}^n$ is called

i) a *(classical) solution* to (1) if $x \in C^k(\mathbb{I}, \mathbb{C}^n)$ and $x$ satisfies (1) pointwise.

ii) a *(classical) solution to the initial value problem* (1)-(2) if $x$ is a solution of (1) and satisfies (2).

We introduce $X_0 := \left[x_0^{(k-1),T} \; \dots \; x_1^T \;\; x_0^T\right]^T \in \mathbb{C}^{kn}$ as an initial vector of the initial value problem consisting of (1) and (2).

iii) An initial vector $X_0$ is called *consistent* to system (1) if the initial value problem (1)–(2) has a solution.

iv) System (1) is called *solvable* if for every sufficiently smooth $f$ and every consistent initial vector $X_0$, the associated initial value problem (1)–(2) has at least one solution. It is called *regular* if it is solvable and the solution is unique.

We recall without proof the following results, see e. g., Theorems 3.9, 3.25 ([3]) .

**Theorem 2 (**[3]**)**. *Let* $E \in C^\ell(\mathbb{I}, \mathbb{R}^{m,n})$, $\ell \in \mathbb{N}_0 \cup \infty$, *with* $\operatorname{rank} E(t) = r$ *for all* $t \in \mathbb{I}$. *Then there exist pointwise unitary functions* $U \in C^\ell(\mathbb{I}, \mathbb{C}^{m,m})$ *and* $V \in C(\mathbb{I}, \mathbb{C}^{n,n})$, *such that* $U^H E V = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix}$, *with pointwise nonsingular* $\Sigma \in C^\ell(\mathbb{I}, \mathbb{C}^{r,r})$.

**Theorem 3 ([3]).** *Let $\mathbb{I} \subset \mathbb{R}$ be a close interval and $M \in C(\mathbb{I}, \mathbb{C}^{m,n})$. Then there exist open intervals $\mathbb{I}_j \subset \mathbb{I}$, $j \in \mathbb{N}$, with* $\overline{\bigcup_{j\in\mathbb{N}} \mathbb{I}_j} = \mathbb{I}, \mathbb{I}_i \cap \mathbb{I}_j = \emptyset$ for $i \neq j$, *and* $r_j \in \mathbb{N}_0$, $j \in \mathbb{N}$ *such that* rank $M(t) = r_j$ for all $t \in \mathbb{I}_j$.

Making use of this theorem, in the remaining part of this research, we can assume constant rank assumptions holds, in particular in Algorithm 13.

For two functions $P \in C(\mathbb{I}, \mathbb{C}^{p,n})$, $Q \in C(\mathbb{I}, \mathbb{C}^{q,n})$, the function pair $(P, Q)$ is said to *have no hidden redundancy* if

$$\operatorname{rank}\left(\begin{bmatrix} P(t) \\ Q(t) \end{bmatrix}\right) = \operatorname{rank}(P(t)) + \operatorname{rank}(Q(t)) \text{ for all } t \in \mathbb{I}.$$

The following lemmas are taken from [17].

**Lemma 4 ([17]).** *Suppose that for $P \in C(\mathbb{I}, \mathbb{C}^{p,n})$, $Q \in C(\mathbb{I}, \mathbb{C}^{q,n})$, the function pair $(P, Q)$ has no hidden redundancy. Let $U \in C(\mathbb{I}, C^{q,q})$ and $V \in C(\mathbb{I}, C^{p,p})$ be two arbitrary functions. Then, the function pair $(UP, VQ)$ has no hidden redundancy.*

If for all $t \in \mathbb{I}$, matrix $\begin{bmatrix} P(t) \\ Q(t) \end{bmatrix}$ is of full row rank then obviously, the function pair $(P, Q)$ has no hidden redundancy. However, the converse is not true as is obvious for $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, since $(P, Q)$ has no hidden redundancy, but $\begin{bmatrix} Q \\ P \end{bmatrix}$ does not have full row rank.

**Lemma 5 ([17]).** *Given two functions $P \in C(\mathbb{I}, \mathbb{C}^{p,n})$, $Q \in C(\mathbb{I}, \mathbb{C}^{q,n})$. Moreover, assume that $P$ has pointwise full row rank, and* rank$(Q)$, rank$\left(\begin{bmatrix} P \\ Q \end{bmatrix}\right)$ *are constants on $\mathbb{I}$, i. e.,* $\operatorname{rank}(Q(t)) = q_1$, $\operatorname{rank}\left(\begin{bmatrix} P(t) \\ Q(t) \end{bmatrix}\right) = q_2$ for all $t \in \mathbb{I}$. *Then, there exists a function* $\begin{bmatrix} S & 0 \\ Z_1 & Z_2 \end{bmatrix} \in C(\mathbb{I}, \mathbb{C}^{p+q})$ *such that the following conditions hold.*

i) $\begin{bmatrix} S \\ Z_1 \end{bmatrix} \in C(\mathbb{I}, \mathbb{C}^{p,p})$ *is pointwise unitary,*

ii) $[Z_1 \quad Z_2] \begin{bmatrix} P \\ Q \end{bmatrix} = 0$,

iii) *the function pair $(SP, Q)$ has no hidden redundancy.*

**Lemma 6.** *Consider $k + 1$ pointwise full row rank functions $R_0 \in C(\mathbb{I}, \mathbb{C}^{r_0,n})$, ..., $R_k \in C(\mathbb{I}, \mathbb{C}^{r_k,n})$, such that none of the function pairs $\left(R_j(t), \left[R_{j-1}^T(t) \quad \dots \quad R_0^T(t)\right]^T\right)$, $j = k, \dots, 1$ has a hidden redundancy. Then, $[R_k^T(t) \quad \dots \quad R_0^T(t)]^T$ has full row rank for all $t \in \mathbb{I}$.*

*Proof.* The proof is straightly followed by induction, i.e.,

$$rank\left(\begin{bmatrix} R_k \\ R_{k-1} \\ \vdots \\ R_0 \end{bmatrix}\right) = rank(R_k) + rank\left(\begin{bmatrix} R_{k-1} \\ \vdots \\ R_0 \end{bmatrix}\right) = rank(R_k) + rank(R_{k-1}) + rank\left(\begin{bmatrix} R_{k-2} \\ \vdots \\ R_0 \end{bmatrix}\right),$$

$$= \dots = \operatorname{rank}(R_k) + \operatorname{rank}(R_{k-1}) + \dots + \operatorname{rank}(R_0),$$

and since $R_k, \dots, R_0$ have pointwise full row rank, then $[R_k^T \quad \dots \quad R_0^T]^T$ has pointwise full row rank. ☐

## 3. Strangeness-index concept and strangeness-free form of high-order DAEs

As stated in the introduction, this section is devoted to a strangeness-index concept and a reformulation algorithm of the high-order linear DAE (1). See also [14]–[16] and the references therein for previous works on this topic. Note that different from previous investigations, we study system (1) by extending the algebraic approach for first order DAEs proposed in [3], [17] and combining with the behavior approach [18]. Let

$$M(t) := [A_k(t) \cdots A_0(t)] \text{ and } X(t) := \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix}, t \in \mathbb{I},\ X_0 := \begin{bmatrix} x^{(k-1)}(t_0) \\ \vdots \\ x(t_0) \end{bmatrix}. \tag{3}$$

Then $M(t)$ (resp., $X(t)$) is called the *behavior function (resp., behavior vector)* of IVP (1)-(2), which can be rewritten as

$$M(t)X(t) = f(t) \text{ for all } t \in \mathbb{I}, \quad X(t_0) = X_0. \tag{4}$$

**Hypothesis 7**. Denote by

$$m_i(t) := \text{rank}([A_k(t), \dots, A_{k-i}(t)]), i = 0, \dots, k,$$

we assume that functions $A_i$, $i = 0, \dots, k$ of system (1) fulfill the following constant rank conditions

$$m_i(t) \equiv m_i \text{ for all } t \in \mathbb{I}, i = 0, \dots, k.$$

Under the Hypothesis 7, we can transform $M$ to the block diagonal form as in the following lemma.

**Lemma 8**. *Consider the behavior function $M$ of system (1). Moreover, suppose that Hypothesis 7 holds. Then, there exists a pointwise nonsingular function $P \in C(\mathbb{I}, \mathbb{C}^{\ell,\ell})$ such that*

$$\widetilde{M} := PM = \left[\begin{array}{c|c|c|c} A_{k,1} & A_{k-1,1} & \dots & A_{0,1} \\ & A_{k-1,2} & \dots & A_{0,2} \\ & & \ddots & \vdots \\ & & & A_{0,k+1} \\ 0 & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_{k+1} \\ v \end{array} \tag{5}$$

*where every function $A_{j,k+1-j}$, $k \geq j \geq 0$ on the main diagonal has pointwise full row rank. Moreover, those ranks are computed via*

$$r_1 = m_0, r_2 = m_1 - m_0, \dots, r_{k+1} = m_k - m_{k-1}, v = \ell - m_k. \tag{6}$$

*Proof*. First, due to Theorem 2, we can find a pointwise nonsingular function $P_1 \in C(\mathbb{I}, \mathbb{C}^{\ell,\ell})$ such that

$$P_1 M = \left[\begin{array}{c|ccc} A_{k,1} & A_{k-1,1} & \dots & A_{0,1} \\ 0 & A_{k-1,2} & \dots & A_{0,2} \end{array}\right], \quad \begin{array}{c} r_1 \\ \ell - r_1 \end{array}$$

and $A_{k,1}$ has pointwise full row rank. Moreover, since

$$\text{rank}(P_1[A_k, A_{k-1}]) = \text{rank}\left(\begin{bmatrix} A_{k,1} & A_{k-1,1} \\ 0 & A_{k-1,2} \end{bmatrix}\right) = m_1, \text{ for all } t \in \mathbb{I},$$

it follows that $\text{rank}\left(A_{k-1,2}(t)\right) = m_1 - m_0 =: r_2$ for all $t \in \mathbb{I}$. Thus, using Theorem 2 again, we compress the 2nd block column from the second block row of $P_1 M$ and then inductively for the other columns of $M$, we determine a sequence of pointwise nonsingular functions $P_i$, $1 \leq i \leq k+1$ such that $P := \prod_1^{i=k+1} P_i$ and $PM$ takes the block diagonal form (5). Furthermore, (6) follows directly from the construction of $P$. ☐

We call the number $r_u := (k+1)r_1 + kr_2 + \cdots + 2r_k + r_{k+1}$ the *upper rank* of the behavior function $M$. Note, that some of the $r_i$ may be zero while the corresponding block row is not present.

In the following, without loss of generality, we assume that the behavior function $M$ is already in the form $\widetilde{M}$. For notational convenience, we omit the variable $t$ in $x$, $f$ and their derivatives, and also in all matrix-valued functions. Rewriting system (4) block row-wise, we obtain the system

$$\begin{aligned} A_{k,1}x^{(k)} + A_{k-1,1}x^{(k-1)} + \ldots + A_{1,1}\dot{x} + A_{0,1}x &= f_1, \\ A_{k-1,2}x^{(k-1)} + \ldots + A_{1,2}\dot{x} + A_{0,2}x &= f_2, \\ &\ldots \\ A_{0,k+1}x &= f_{k+1}, \\ 0 &= f_{k+2}. \end{aligned} \tag{7}$$

Recall that the diagonal blocks $A_{k,1}, A_{k-1,2}, \dots, A_{0,k+1}$ have pointwise full row rank, therefore in system (7), for every $j$ with $k \geq j \geq 0$, the $(k+1-j)$-th block row

$$A_{j,k+1-j}x^{(j)} + \cdots + A_{0,k+1-j}x = f_{k+1-j},$$

represents $r_{k+1-j} = \operatorname{rank}(A_{j,k+1-j})$ scalar differential equations of order $j$. We apply the algebraic approach to reduce the number of scalar differential equations of order $j$, i. e., by using differential equations of order smaller than $j$ and their derivatives. Let us illustrate this idea for the case $j = k$ in the Lemma 9 below. For notational convenience, by $L(x, \dots, x^{(j)})$ we denote an unspecified linear function of $x, \dots, x^{(j)}$, $0 \leq j \leq k$. First we need to assume that functions $\begin{bmatrix} A_{k,1} \\ A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}$, $\begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}$ have constant rank on $\mathbb{I}$. By dividing $\mathbb{I}$ into a union of (at most) countable disjoint intervals $\overline{\bigcup_{j\in\mathbb{N}} \mathbb{I}_j}$ as in Theorem 3, this constant rank assumption will be fulfilled on each interval $\mathbb{I}_j$.

Suppose that the function pair $\left(A_{k,1}, \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}\right)$ has a hidden redundancy, using Lemma 5, we can find functions $S_k, Z_{k,k}, Z_{k,k-1}, \dots, Z_{k,0}$ of appropriate size such that

i) $\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix} \in C(\mathbb{I}, \mathbb{C}^{r_1,r_1})$ is pointwise unitary,

ii) $\sum_0^{i=k} Z_{k,i}\, A_{i,k+1-i} = 0,$ (8)

iii) the function pair $\left(S_k A_{k,1}, \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}\right)$ has no hidden redundancy.

The following lemma shows that we can reduce the number of scalars $k$-th order differential equations in system (7) from $r_1 = \operatorname{rank}(A_{k,1})$ to $d_1 = \operatorname{rank}(S_k A_{k,1})$.

**Lemma 9**. *The system (7) has the same solution set as the DAE*

$$
\begin{aligned}
&S_k A_{k,1} x^{(k)} + S_k A_{k-1,1} x^{(k-1)} + \ldots + S_k A_{1,1} \dot{x} + S_k A_{0,1} x = S_k f_1, \\
&L(x, \ldots, x^{(k-1)}) = \sum_{i=0}^{k} Z_{k,i} f_{k+1-i}^{(k-i)}, \\
&A_{k-1,2} x^{(k-1)} + \ldots + A_{1,2} \dot{x} + A_{0,2} x = f_2, \\
&\ldots \\
&A_{0,k+1} x = f_{k+1}, \\
&0 = f_{k+2}.
\end{aligned} \tag{9}
$$

*for some linear function L of* $x, \ldots, x^{(k-1)}$.

*Proof.* Firstly, we replace the first equation of system (7) by the scaled one

$$\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix} \left( A_{k,1} x^{(k)} + A_{k-1,1} x^{(k-1)} + \cdots + A_{1,1} \dot{x} + A_{0,1} x \right) = \begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix} f_1,$$

or equivalently,

$$
\begin{aligned}
&S_k A_{k,1} x^{(k)} + S_k A_{k-1,1} x^{(k-1)} + \cdots + S_k A_{1,1} \dot{x} + S_k A_{0,1} x = S_k f_1, \\
&Z_{k,k}\, A_{k,1}\, x^{(k)} + Z_{k,k} \left( A_{k-1,1} x^{(k-1)} + \cdots + A_{0,1} x \right) = Z_{k,k} f_1.
\end{aligned} \tag{10}
$$

then obviously, it does not change the solution set. On the other hand, for $1 \le i \le k-1$, the $(k+1-i)$-th equation of (7) is

$$A_{i,k+1-i} x^{(i)} + \sum_{q=0}^{i-1} A_{q,k+1-i}\, x^{(q)} = f_{k+1-i}.$$

Taking the $(k-i)$-th derivative of it and scaling the resulting equation with $Z_{k,i}$, we obtain

$$Z_{k,i} \left( \frac{d}{dt} \right)^{k-i} \left( A_{i,k+1-i} x^{(i)} + \sum_{q=0}^{i-1} A_{q,k+1-i}\, x^{(q)} \right) = Z_{k,i} f_{k+1-i}^{(k-i)},$$

$$\Leftrightarrow\ Z_{k,i} A_{i,k+1-i} x^{(k)} + \underbrace{\sum_{p=0}^{k-i-1} \binom{k-i}{p} A_{i,k+1-i}^{(k-i-p)} x^{(i+p)} + Z_{k,i} \frac{d^{k-i}}{dt^{k-i}} \left( \sum_{q=0}^{i-1} A_{q,k+1-i}\, x^{(q)} \right)}_{=:\, L(x, \ldots, x^{(k-1)})} = Z_{k,i} f_{k+1-i}^{(k-i)}.$$

Thus, we have the new equation

$$Z_{k,i} A_{i,k+1-i} x^{(k)} + L\left(x, \ldots, x^{(k-1)}\right) = Z_{k,i} f_{k+1-i}^{(k-i)}. \tag{11}$$

Replacing equation (10) in system (7) by the sum of itself and all equations of the type (11), $i = 0, \ldots, k-1$, we get

$$\sum_{0}^{i=k} Z_{k,i}\, A_{i,k+1-i} x^{(k)} + L\left(x, \ldots, x^{(k-1)}\right) = \sum_{i=0}^{k} Z_{k,i}\, f_{k+1-i}^{(k-i)},$$

and hence, the equality (8) implies that $L\left(x, \ldots, x^{(k-1)}\right) = \sum_{i=1}^{k} Z_{k,i}\, f_{k+1-i}^{(k-i)}$. ☐

Note that, in order to apply the same argument for the block rows numbered $j = k-1, \ldots, 1$, in system (7), it is necessary to require the constant rank conditions in the following hypothesis.

**Hypothesis 10**. Denote by

$$q_j(t) = \operatorname{rank}\left(\begin{bmatrix} A_{j,k+1-j}(t) \\ A_{j-1,k+2-j}(t) \\ \vdots \\ A_{0,k+1}(t) \end{bmatrix}\right), j = k, \dots, 1, t \in \mathbb{I},$$

we assume that

$$q_j(t) \equiv q_j \text{ for all } t \in \mathbb{I}, j = k, \dots, 1.$$

Applying the same argument to the block rows numbered $j = k-1, \dots, 1$, we obtain the following two lemmas.

**Lemma 11**. *Consider the DAE (1) and its behavior form (4). Moreover, assume that Hypotheses 7, 10 hold. Then, there exist functions $S_j$, $Z_{j,i}$, $j = k, \dots, 1$, $i = j, \dots, 0$ of appropriate size such that the following assertions hold true*

i) *the functions* $\begin{bmatrix} S_j \\ Z_{j,j} \end{bmatrix} \in C(\mathbb{I}, \mathbb{C}^{r_j, r_j})$, $k \geq j \geq 1$ *are pointwise unitary,*

ii) *for each j with* $k \geq j \geq 1$, $Z_{j,j} A_{j,k+1-j} + [Z_{j,j-1} \dots Z_{j,0}] \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} = 0$,

iii) *for each j with* $k \geq j \geq 1$, *the function pair* $\left(S_j A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix}\right)$ *has no hidden redundancy.*

*Proof*. For each $j$ with $k \geq j \geq 1$, by applying Lemma 5 to the function pair

$$\left(A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix}\right)$$

we obtain functions $S_j$, $Z_{j,i}$, $i = j, \dots, 0$ that satisfy conditions i)-iii). ☐

Setting

$$\tilde{P} := \operatorname{diag}\left(\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}, \dots, \begin{bmatrix} S_1 \\ Z_{1,1} \end{bmatrix}, I_{r_{k+1}+v}\right) \in C\left(\mathbb{I}, \mathbb{C}^{\ell,\ell}\right),$$

and scaling system (1) with $\tilde{P}$ from the left we obtain

$$\begin{bmatrix} S_k A_{k,1} & | & * & | & \dots & | & * \\ Z_{k,k} A_{k,1} & | & * & | & \dots & | & * \\ & | & S_{k-1} A_{k-1,2} & | & \dots & | & * \\ & | & Z_{k-1,k-1} A_{k-1,2} & | & \dots & | & * \\ & | & & | & \ddots & | & \vdots \\ & | & & | & & | & A_{0,k+1} \\ 0 & | & 0 & | & \dots & | & 0 \end{bmatrix} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ Z_{k,k} f_1 \\ S_{k-1} f_2 \\ Z_{k-1,k-1} f_2 \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}. \tag{12}$$

For each $j$ with $k \geq j \geq 1$, we then reduce the number of scalar differential equations of order $j$ by eliminating the block $Z_{j,j} A_{j,k+1-j}$ of (12), as in the following lemma.

**Lemma 12**. *Let functions $S_j$, $Z_{j,i}$, $j = k, \dots, 1$, $i = j, \dots, 0$, be defined as in Lemma 11. Then, the DAE (12) has the same solution set as the DAE*

$$\begin{matrix} d_1 \\ s_1 \\ d_2 \\ s_2 \\ \vdots \\ d_{k+1} \\ v \end{matrix} \underbrace{\begin{bmatrix} S_k A_{k,1} & | & * & | & \dots & | & * \\ 0 & | & * & | & \dots & | & * \\ & | & S_{k-1}A_{k-1,2} & | & \dots & | & * \\ & | & 0 & | & \dots & | & * \\ & | & & | & \ddots & | & \vdots \\ & | & & | & & | & A_{0,k+1} \\ 0 & | & 0 & | & \dots & | & 0 \end{bmatrix}}_{\breve{M}} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} S_k f_1 \\ g_{2k} \\ S_{k-1} f_2 \\ g_{2(k-1)} \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}}_{\breve{f}}, \quad (13)$$

*where* $g_{2j} := \sum_{i=0}^{j} Z_{j,i} f_{k+1-i}^{(j-i)}, \quad j = k, \dots, 1.$

*Proof.* The form (13) can be directly obtained by applying the same argument as in Lemma 9 for $j = k, \dots, 1$. ▢

From (13), we deduce that $r_j = d_j + s_j$, $j = 1, \dots, k+1$, $s_{k+1} = 0$ and therefore the upper rank of the behavior function $\breve{M}$ can be estimated via

$$\begin{aligned} \breve{r}_u &\le (k+1)d_1 + k(s_1 + d_2) + \dots + (s_k + d_{k+1}), \\ &= (k+1)(d_1 + s_1) + k(d_2 + s_2) + \dots + (d_{k+1} + s_{k+1}) - \sum_{i=0}^{k} s_i, \\ &= (k+1)r_1 + kr_2 + \dots + r_{k+1} - \sum_{i=0}^{k} s_i = r_u - \sum_{i=0}^{k} s_i. \end{aligned}$$

This reduction of the upper rank leads to the following algorithm.

**Algorithm 13**.

**Input:** The DAE (1) and its behavior form (4).

**Begin:** Set $\alpha = 0$ and let $M^0 = M$, $f^0 = f$,

**Step 1.** Determine a pointwise nonsingular function $P \in C(\mathbb{I}, \mathbb{C}^{\ell,\ell})$ (as in Lemma 8) such that

$$PM^{\alpha} = \begin{bmatrix} A_{k,1} & | & A_{k-1,1} & | & \dots & | & A_{0,1} \\ & | & A_{k-1,2} & | & \dots & | & A_{0,2} \\ & | & & | & \ddots & | & \vdots \\ & | & & | & & | & A_{0,k+1} \\ 0 & | & 0 & | & \dots & | & 0 \end{bmatrix}, \quad \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_{k+1} \\ v \end{matrix}$$

where all the functions on the main diagonal have pointwise full row rank, and let

$$r_u^{\alpha} := (k+1)r_1 + mr_2 + \dots + 2r_k + r_{k+1},$$

be the upper rank of the behavior function $M^{\alpha}$ in the $\alpha$-th iteration.

**Step 2.** Determine functions $S_j$, $Z_{j,i}$, $j = k, \dots, 1$, $i = j, \dots, 0$ of appropriate sizes such that for each $k \ge j \ge 1$ the following conditions hold

i) the function $\begin{bmatrix} S_j \\ Z_{j,j} \end{bmatrix} \in C(\mathbb{I}, \mathbb{C}^{r_j, r_j})$ is pointwise unitary,

ii) $Z_{j,j} A_{j,k+1-j} + \begin{bmatrix} Z_{j,j-1} \dots Z_{j,0} \end{bmatrix} \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} = 0,$

iii) the function pair $\left( S_j A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} \right)$ has no hidden redundancy.

**Step 3.** Setting

$$\tilde{P} := \operatorname{diag}\left( \begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}, \ldots, \begin{bmatrix} S_1 \\ Z_{1,1} \end{bmatrix}, I_{r_{k+1}+v} \right) \in C(\mathbb{I}, \mathbb{C}^{\ell,\ell}),$$

and scaling system (1) with $\tilde{P}$ from the left we obtain

$$\begin{bmatrix} S_k A_{k,1} & | & * & | & \ldots & | & * \\ Z_{k,k} A_{k,1} & | & * & | & \ldots & | & * \\ & | & S_{k-1} A_{k-1,2} & | & \ldots & | & * \\ & | & Z_{k-1,k-1} A_{k-1,2} & | & \ldots & | & * \\ & | & & | & \ddots & | & \vdots \\ & | & & | & & | & A_{0,k+1} \\ 0 & | & 0 & | & \ldots & | & 0 \end{bmatrix} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ Z_{k,k} f_1 \\ S_{k-1} f_2 \\ Z_{k-1,k-1} f_2 \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}. \tag{14}$$

**Step 4.** For each $k \geq j \geq 1$, we then reduce the number of scalar differential equations of order $j$ by eliminating the block $Z_{j,j} A_{j,k+1-j}$ of (14), as in Lemma 12. In this way, we obtain the system

$$\begin{matrix} d_1 \\ s_1 \\ d_2 \\ s_2 \\ \vdots \\ d_{k+1} \\ v \end{matrix} \underbrace{\begin{bmatrix} S_k A_{k,1} & | & * & | & \ldots & | & * \\ 0 & | & * & | & \ldots & | & * \\ & | & S_{k-1} A_{k-1,2} & | & \ldots & | & * \\ & | & 0 & | & \ldots & | & * \\ & | & & | & \ddots & | & \vdots \\ & | & & | & & | & A_{0,k+1} \\ 0 & | & 0 & | & \ldots & | & 0 \end{bmatrix}}_{\breve{M}} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} S_k f_1 \\ g_{2k} \\ S_{k-1} f_2 \\ g_{2(k-1)} \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}}_{\breve{f}},$$

with $g_{2j} := \sum_{i=0}^{j} Z_{j,i} f_{k+1-i}^{(j-i)}, \quad j = k, \ldots, 1.$

Let $s^\alpha := \sum_{i=0}^{k} s_i$ . If $s^\alpha = 0$ then we terminate the process here. Otherwise, we then increase $\alpha$ by 1, set $M^\alpha = \breve{M}$, $f^\alpha = \breve{f}$, and repeat the process from Step 1.

**End.**

In Algorithm 13, we construct a decreasing sequence $\left\{ r_u^\alpha \right\}_{\alpha \in \mathbb{N}}$ which satisfies $r_u^{\alpha+1} \leq r_u^\alpha - s^\alpha$, where $s^\alpha = \sum_{i=0}^{k} s_i \geq 0$. Since this squence $\left\{ r_u^\alpha \right\}_{\alpha \in \mathbb{N}}$ is non-negative, Algorithm 13 must terminate after a finite number of iterations.

**Definition 14**. Consider the DAE (1) and the sequence $(r_u^\alpha, s^\alpha)$, $\alpha \in \mathbb{N}$ of characteristic invariants generated by Algorithm 13. Then, we call

$$\mu = \min\{\alpha \in \mathbb{N}^0 |\ s^\alpha = 0\}$$

the *strangeness index* of (1).

Note that, the strangeness-index $\mu$ is well-defined if and only if Hypotheses 7, 10 are satisfied in every iteration of Algorithm 13.

**Theorem 15**. *Consider the DAE (1) and assume that the strangeness-index $\mu$ is well-defined. Then, the DAE (1) has the same solution set as the so-called strangeness-free DAE*

$$\begin{bmatrix} \hat{A}_{k,1} & | & \hat{A}_{k-1,1} & | & \dots & | & \hat{A}_{0,1} \\ & | & \hat{A}_{k-1,2} & | & \dots & | & \hat{A}_{0,2} \\ & | & & | & \ddots & | & \vdots \\ & | & & | & & | & \hat{A}_{0,k+1} \\ 0 & | & 0 & | & \dots & | & 0 \end{bmatrix} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k+1} \\ \hat{f}_{k+2} \end{bmatrix}, \tag{15}$$

*where* $\begin{bmatrix} \hat{A}_{k,1}^T & \dots & \hat{A}_{0,k+1}^T \end{bmatrix}^T$ *has pointwise full row rank.*

*Proof*. Clearly, after carrying out Algorithm 13, we obtain a system of the form (15), where $\hat{A}_{k,1}, \dots, \hat{A}_{0,k+1}$ have pointwise full row rank and none of the function pairs

$$\left( \hat{A}_{i,k+1-i}, \begin{bmatrix} \hat{A}_{i-1,k+2-i} \\ \vdots \\ hA_{0,k+1} \end{bmatrix} \right), \qquad i = k, \dots, 1,$$

has a hidden redundancy. Applying Lemma 6 to the functions $\hat{A}_{i,k+1-i}$, $i = k, \dots, 0$, it follows that $\begin{bmatrix} \hat{A}_{k,1}^T & \dots & \hat{A}_{0,k+1}^T \end{bmatrix}^T$ has pointwise full row rank. ☐

Obviously, if at $t = 0$ the consistency assumptions

$$\begin{cases} \left(\dfrac{d}{dt}\right)^i \left( \hat{A}_{k-1,2} x^{(k-1)}(t) + \ldots + \hat{A}_{1,2} x^{(1)}(t) + \hat{A}_{0,2} x(t) - \hat{f}_2(t) \right) = 0, \ i = 0,1, \\ \dots \\ \left(\dfrac{d}{dt}\right)^i \left( \hat{A}_{0,k+1} x(t) - \hat{f}_{k+1}(t) \right) = 0, \ i = 0, \ldots, k, \\ \hat{f}_{k+2}(t) = 0, \end{cases} \tag{16}$$

hold, then we can differentiate all except the first equation of system (15) to obtain an underlying ODE as in the next theorem.

**Theorem 16**. *Consider the DAE (1) and assume that the strangeness-index $\mu$ is well-defined. Moreover, suppose that the consistency condition (16) is satisfied. Then, (1) has the same solution set as the underlying ODE*

$$\begin{bmatrix} \hat{A}_{k,1} & | & * & | & \dots & | & * & | & * \\ \hat{A}_{k-1,2} & | & * & | & \dots & | & * & | & * \\ \vdots & | & * & | & \dots & | & * & | & * \\ \hat{A}_{1,k} & | & * & | & \dots & | & * & | & * \\ \hat{A}_{0,k+1} & | & * & | & \dots & | & * & | & * \end{bmatrix} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x^{(1)} \\ x \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2^{(1)} \\ \vdots \\ \hat{f}_k^{(k-1)} \\ \hat{f}_{k+1}^{(k)} \end{bmatrix},$$

*where the function* $\begin{bmatrix} \hat{A}_{k,1}^T & \dots & \hat{A}_{0,k+1}^T \end{bmatrix}^T$ *has pointwise full row rank.*

The following corollary is a direct consequence of Theorem 15.

**Corollary 17**. Consider the initial value-problem (1)–(2), and let (15) be the strangeness-free formulation of the DAE (1). Then we have:

i) The DAE (1) is solvable if and only if the following consistency condition holds

$$0 = \hat{f}_{k+2}(t) \text{ for all } t \in \mathbb{I}.$$

ii) The initial vector $X_0$ is consistent if and only if

$$\begin{bmatrix} \hat{A}_{k-1,2} & \hat{A}_{k-2,2} & \cdots & \hat{A}_{0,2} \\ & \hat{A}_{k-2,3} & \cdots & \hat{A}_{0,3} \\ & & \ddots & \vdots \\ & & & \hat{A}_{0,k+1} \end{bmatrix} \begin{bmatrix} x_0^{(k-1)} \\ x_0^{(k-2)} \\ \vdots \\ x_0 \end{bmatrix} = \begin{bmatrix} \hat{f}_2(0) \\ \hat{f}_3(0) \\ \vdots \\ \hat{f}_{k+1}(0) \end{bmatrix}.$$

iii) To guarantee that at least $x \in C^k(\mathbb{I}, \mathbb{C}^{n,n})$, $f$ must satisfies that $f \in C^{k\mu+k}(\mathbb{I}, \mathbb{C}^{\ell})$.

iv) The corresponding initial value problem (1)-(2) is regular if and only if in addition, the function $[\hat{A}_{k,1}^T \quad \ldots \quad \hat{A}_{0,k+1}^T]^T$ is square.

## 4. Examples

In this section we illustrate our results by considering some examples.

**Example 1**. We consider the second order DAE from

$$\begin{bmatrix} 1 & t+1 \\ t & t^2+t \end{bmatrix} \ddot{x} + \begin{bmatrix} 0 & 2 \\ 0 & 2t \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & t \\ 1+t & 1+t+t^2 \end{bmatrix} x = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad t \geq 0. \tag{17}$$

The system in behavior form is

$$\underbrace{\begin{bmatrix} 1 & t+1 & | & 0 & 2 & | & 1 & t \\ t & t^2+t & | & 0 & 2t & | & 1+t & 1+t+t^2 \\ & & & & & & & \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x} \\ \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Scaling $M$ with $\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$, we bring $M$ to the block diagonal form

$$\widetilde{M}_0 = \begin{bmatrix} 1 & t+1 & | & 0 & 2 & | & 1 & t \\ 0 & 0 & | & 0 & 0 & | & 1 & 1+t \end{bmatrix} =: \begin{bmatrix} A_{21} & A_{11} & A_{01} \\ 0 & 0 & A_{03} \end{bmatrix}. \quad \begin{matrix} r_1 = 1 \\ r_3 = 1 \end{matrix}$$

All the constant rank conditions are satisfied since

$$\operatorname{rank}(A_2) = 1, \operatorname{rank}([A_2 \quad A_1]) = 1, \operatorname{rank}([A_2 \quad A_1 \quad A_0]) = 2,$$
$$\operatorname{rank}(A_{21}) = 1, \operatorname{rank}(A_{03}) = 1, \operatorname{rank}\left(\begin{bmatrix} A_{21} \\ A_{03} \end{bmatrix}\right) = 1.$$

Obviously, the function pair $(A_{21}, A_{03})$ has a hidden consistency, we therefore perform Algorithm 13 which contains only one iteration (index reduction step) with $S_1 = \emptyset$, $Z_{11} = 1$, $Z_{12} = 0$, $Z_{13} = -1$. Finally, we obtain the strangeness-free formulation

$$\begin{bmatrix} 0 & 0 & | & 0 & 2 & | & 1 & t \\ 0 & 0 & | & 0 & 0 & | & 1 & 1+t \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} f_1 - \ddot{f}_2 \\ f_2 \end{bmatrix}.$$

Since only one index reduction step is used, the strangeness-index is $\mu = 1$.

**Example 2**. Consider the model of a two dimensional, three link mobile manipulator from [19]–[21]. The linearized equation around a non-stationary solution yields a linear time varying model in 3D of the form

$$\begin{bmatrix} M(t) & 0 \\ 0 & 0 \end{bmatrix} \ddot{x} + \begin{bmatrix} D(t) & 0 \\ 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} K(t) & -F^T(t) \\ F(t) & 0 \end{bmatrix} x = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

Moreover, the matrix-value functions $M$, $D$, $K$ are pointwise positive definite, and $F(t)$ has pointwise full row rank. The index reduction algorithm of the system above implies that the strangeness index is $\mu = 2$.

**Example 3.** Consider a simple electrical circuit that includes a resistor $(R)$, an inductor $(L)$, a capacitor $(C)$, with an external force input $(u(t))$, a current source $(f_2(t))$, and an additional forcing term $(f_1(t))$, Figure 1. Here we assume that the resistance R, inductance L, capacitance C are functions of time t.

Here $i_1$ and $i_2$ are the currents flowing through different parts of the circuit. Specifically, $i_1$ is the current flowing through the resistor (R) and the inductor (L), and $i_2$ is the current flowing through the capacitor (C). Besides that, $v_1$, $v_2$, and $v_3$ are voltages at different points in the circuit. Here $v_1$ (or u(t)) is the voltage of the source connected to the resistor (R), $v_2$ is the voltage across the inductor (L), and $v_3$ is the voltage across the capacitor (C). Making use of the Kirchhoff's voltage law, which states that the sum of the electrical potential differences (voltages) around any closed loop or mesh in a network is always equal to zero, we obtain the following equations for the circuit

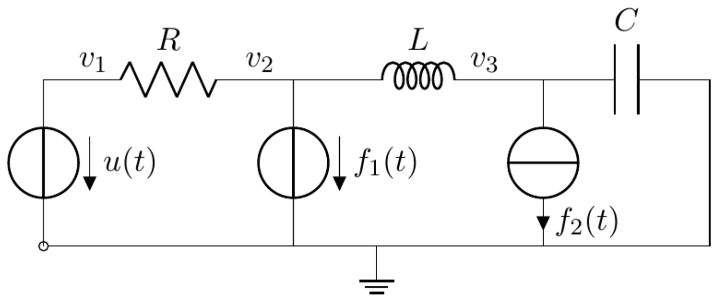


**Figure 1**. Second order electrical circuit.

$$\begin{aligned} Ri_1 + L\frac{di_1}{dt} + \frac{1}{C}v_3 &= -u(t) + f_1(t) \\ L\frac{di_2}{dt} + Ri_2 + v_2 &= 0 \\ \frac{1}{C}v_3 + v_2 &= f_2(t) \end{aligned} \tag{18}$$

Substitute the capacitor and inductor equations

$$i_1 = C\frac{dv_3}{dt}, \quad i_2 = \frac{dv_2}{dt}$$

into system (18) we then have a second order system

$$\begin{bmatrix} LC & 0 \\ 0 & L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_3'' \\ v_2'' \end{bmatrix} + \begin{bmatrix} RC + LC' & 0 \\ 0 & R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_3' \\ v_2' \end{bmatrix} + \begin{bmatrix} \frac{1}{C} & 0 \\ 0 & 1 \\ \frac{1}{C} & 1 \end{bmatrix} \begin{bmatrix} v_3 \\ v_2 \end{bmatrix} = \begin{bmatrix} -u + f_1 \\ 0 \\ f_2 \end{bmatrix} \tag{19}$$

Now in the behavior form we have that

$$M := \left[\begin{array}{cc|c|ccc} LC & 0 & RC + LC' & 0 & C^{-1} & 0 \\ 0 & L & 0 & R & 0 & 1 \\ 0 & 0 & 0 & 0 & C^{-1} & 1 \end{array}\right] =: \begin{bmatrix} A_{21} & A_{11} & A_{01} \\ 0 & A_{13} & A_{03} \end{bmatrix}, \quad \begin{matrix} r_1 = 2 \\ r_3 = 1 \end{matrix}$$

Here we see that clearly the function pair $(A_{21}, A_{03})$ has a hidden consistency, we therefore perform Algorithm 13, which contains only one iteration (index reduction step). This implies that the strangeness index is $\mu = 1$.

## 5. Conclusion

In this study, we have delved into the solvability analysis of linear differential-algebraic equations (DAEs) with time-varying coefficients. We have successfully extended the strangeness-index concept, originally proposed in [3], to systems of arbitrarily high order by integrating the algebraic and behavior approaches [3], [17], [18]. This extension of the strangeness-index concept mirrors the approach taken for first-order DAEs, necessitating certain constant rank assumptions at each stage of the index reduction process (Algorithm 13). The successful application of this algorithm yields not only an underlying ordinary differential equation (ODE) but also a consistency condition for the initial vector. Looking ahead, we see the exploration of derivative arrays for high-order DAEs, particularly from a numerical perspective, as a promising avenue for future research.

**Appendix**

*Nomenclature*

| | |
|---|---|
| $\mathbb{N}$ | set of natural numbers including 0 |
| $\mathbb{R}$ ($\mathbb{C}$) | set of real (complex) numbers |
| $\mathbb{I} = [t_0, t_f)$ | time interval |
| $\mathbb{R}^{\ell,n}$ ($\mathbb{C}^{\ell,n}$) | space of real (complex) matrices of size $\ell \times n$ |
| $I$ ($I_n$) | identity matrix (of size $n \times n$) |
| $x^{(j)}$ | the $j^{th}$-derivative of a vector-valued function $x(t)$ |
| $x^{(j),T}$ | transpose of $x^{(j)}$ |
| $C^k(\mathbb{I}, \mathbb{C}^n)$ | space of $k$-time continuously differentiable functions from $\mathbb{I}$ to $\mathbb{C}^n$ |
| $\operatorname{rank}(\cdot)$ | rank of a matrix or a matrix-valued function |
| $r_u$ | upper rank of a matrix of a matrix-valued function |
| $L(x, \dots, x^{(j)})$ | unspecified linear function of $x, \dots, x^{(j)}$. |
| $*$ | unspecified matrices or matrix-valued functions. |

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