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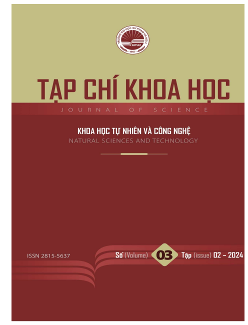

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A note on the existence of solutions to the semi-affine variational inequalities problems

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Abstract

The semi-affine variational inequality problem offers a general and versatile framework applicable to many problems in economics, mathematical physics, operations research, and mathematical programming. One of the important applications of the semi-affine variational inequality problem is quadratic programming. It is well-known that the first-order necessary optimality condition for a constrained optimization problem can be rewritten as a variational inequality. This paper investigates the existence of solutions for the semi-affine variational inequality problem in the finite-dimensional Hilbert spaces. Under suitable conditions, we show that the solution set of the semi-affine variational inequality problem is nonempty. The obtained results contribute to and complement the existing literature.

Keywords: Semi-affine variational inequalities, finite-dimensional Hilbert space, solution existence, recession cone, normal cone

1. Introduction

Let $\mathcal{H}$ be a real finite-dimensional Hilbert space and let K be a nonempty closed convex set in $\mathcal{H}$. In this work, we will address the *semi-affine variational inequality problem*, denotes (sAVI(T,c,K)),

$$\begin{cases} \text{find } x \in K \\ \text{such that } \langle Tx + c, y - x \rangle \geq 0 \quad \forall y \in K, \end{cases} \qquad \text{(sAVI(T,c,K))}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathcal{H}$, T is a bounded linear operator on $\mathcal{H}$ and $c \in \mathcal{H}$. The solution set of (sAVI(T, c, K)) is denoted by Sol(sAVI(T, c, K)).

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The problem (sAVI(T,c,K)) is a natural generalization of the (classical) affine variational inequality problem, which was introduced by Tam in [1]. The generalization here is to use a convex set in $\mathcal{H}$ other than the polyhedral convex set in $\mathcal{H}$ - the setting for the affine variational inequality problem which has been extensively studied in [2]–[5] and references therein. Problem (sAVI(T,c,K)) also arises under optimal conditions (see e.g., [6], [7] and references therein). For the problem (sAVI(T,c,K)) in Euclidean spaces, stability results have been explored, as discussed in [1]. These results encompass the boundedness and stability of solutions under arbitrary perturbations of sufficiently small magnitude. The upper and lower semicontinuity of the solution mapping were discussed in [7] by Nghi, particularly in the case where K is defined by finitely many convex quadratic constraints.

An extension of (sAVI(T,c,K)) is a problem of the form: finding an element $x \in \mathbb{R}^n$ such that

$$G(x) \in K \text{ and } \langle F(x), y - G(x)\rangle \geq 0, \forall y \in K, \tag{1}$$

where $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are two given continuous maps. In [8], Tam and Nghi provided results for the existence of solutions to problem (1). However, when applied to the problem (sAVI(T,c,K)), the conditions for the solution existence given in [8] are equivalent to assuming either the set K is compact or T is strictly monotone. Since the semi-affine variational inequality is a subclass of variational inequality, the solution existence results from variational inequality (see, e.g., [9]–[11] and the references therein) can be applied to the semi-affine variational inequality. However, due to its special structure, we can derive more consistent results for that problem. The primary objective of this work is to establish a set of conditions under which the semi-affine variational inequality problem has a solution.

The remainder of the paper is organized as follows. In Section 2, some notions and results which are useful in the sequel are presented. In Section 3, we prove, under suitable conditions, the solution set of the generalized affine variational inequality is nonempty. Section 4 presents several examples. Finally, we conclude our paper by emphasizing the results that have been obtained.

2. Notations and preliminary results

Throughout this paper, $\mathcal{H}$ denotes an infinite-dimensional Hilbert space equipped with the scalar product $\langle \cdot,\cdot \rangle$ and the induced norm $\|\cdot\|$. For a nonempty, closed, convex set $K \subset \mathcal{H}$, the *recession cone* of K, denoted by 0^+K, is defined as:

$$0^+K = \{v \in \mathcal{H} \mid x + tv \in K \quad \forall x \in K \quad \forall t \geqslant 0\}.$$

If $K \subset \mathcal{H}$ is a nonempty set and $x \in clK$ (the closure of K), then *normal cone* (see [12], [13] and [14]) of K at x is given by:

$$N_K(x) = \{u \in \mathcal{H} \mid \langle u, y - x\rangle \leq 0 \text{ for all } y \in K\}.$$

Definition 2.1. (cf.[15] and [16]) Let C be a closed convex cone in Hilbert space $\mathcal{H}$ and let T be bounded linear operator on $\mathcal{H}$. We say that

(i) T is positive semidefinite on C if $\langle Tv, v\rangle \geq 0$ for all $v \in C$;

(ii) T is positive semidefinite plus on C if T is positive semidefinite on C and if

$$v \in C, \langle Tv, v\rangle = 0 \text{ then } (T + T^*)v = 0;$$

(iii) T is positive on C if $\langle Tv, v\rangle > 0$ for all $v \in C$, $v \neq 0$;

(iv) T is coercive on C if there is an $\alpha > 0$ such that $\langle Tv, v\rangle > \alpha \| v\|^2$ for all $v \in C$.

Let $\overline{x}$ be a point belonging to K. For any number r, we denote,

$$\overline{\mathcal{B}} := \{x \in \mathcal{H} \mid \| x - \overline{x} \| \leqslant r\} \text{ and } K_r := K \cap \overline{\mathcal{B}}.$$

Lemma 2.1. *Let $\overline{x}$ be a point belonging to K. The problem* (sAVI(T,c,K)) *has a solution if and only if there exists some $r > 0$ so that the variational inequality problem* $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$ *has a solution x_r with $\| x_r - \overline{x} \| < r$.*

Proof. This proof is similar to the proof of [9, Theorem 4.2]. □

Lemma 2.2. *Let $\overline{x}$ be a point belonging to K. Suppose that there exists $r > 0$ such that K_r is nonempty set. Then, x_r is a solution to the variational inequality problem* $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$ *if and only if, there exist some scalar μ_r, $\mu_r \geq 0$, such that*

$$-Tx_r - c - \mu_r(x_r - \overline{x}) \in N_k(x_r). \tag{2}$$

Proof. We have

$$\begin{aligned} x_r \in \mathrm{Sol}(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r})) &\Leftrightarrow \langle Tx_r + c, x - x_r\rangle \geq 0 \quad \forall x \in K_r \\ &\Leftrightarrow \langle -Tx_r - c, x - x_r\rangle \leq 0 \quad \forall x \in K_r \\ &\Leftrightarrow -Tx_r - c \in N_{K_r}(x_r). \end{aligned} \tag{3}$$

Since $K_r = K \cap (\mathrm{int}\mathcal{B}) \neq \varnothing$, by [17, Theorem 3.10],

$$N_{K_r}(x_r) = N_{\overline{\mathcal{B}}}(x_r) + N_K(x_r).$$

Hence

$$x_r \in \mathrm{Sol}(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r})) \Leftrightarrow -Tx_r - c \in \left(N_{\overline{\mathcal{B}}}(x_r) + N_K(x_r)\right).$$

It is easy to check that $N_{\overline{\mathcal{B}}}(x_r) = \{\mu_r(x_r - \overline{x})\}$ for all $\mu_r \geq 0$. Consequently, x_r is a solution of the problem $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$ if and only if there exists some scalar $\mu_r \geq 0$ such that

$$-Tx_r - c - \mu_r(x_r - \overline{x}) \in N_K(x_r),$$

and the proof is complete. □

Note that if condition (2) is satisfied for all $r > 0$, then the (sAVI(T,c,K)) problem is uncertain to have a solution.

Lemma 2.3. *Consider the problem* (sAVI(T,c,K)). *Suppose that $\overline{x} \in K$. Then, $\overline{x}$ is a solution to the variational inequality problem* (sAVI(T,c,K)) *if and only if*

$$-T\overline{x} - c \in N_K(\overline{x}).$$

Proof. This proof is similar to the proof of Lemma 2.2. □

To prove our main result, we need to use the concept of an exceptional family of elements, as introduced by G. Isac et al. in [18].

Definition 2.2. We say that $\{x_r\}_{r>0} \subset K$ is an exceptional family for the variational inequality (sAVI(T,c,K)), if the following conditions are satisfied:

(i) $\| x_r \| \to \infty$ as $r \to +\infty$,

(ii) for any $r > 0$ there exists a real number $\mu_r \geq 0$ such that

$$-Tx_r - c - \mu_r (x_r - \overline{x}) \in N_K (x_r).$$

3. Main results

In order to prove the solution existence of (sAVI(T,c,K)), we need the following lemmas.

Lemma 3.1 (cf. [19]). *Let K be a nonempty closed convex set in $\mathcal{H}$. Suppose that $x^k \in K$ for all k, $\| x^k \| \to \infty$ as $k \to \infty$ and $\| x^k \|^{-1} x^k$ weakly converges to $\overline{v}$. Then, $\overline{v} \in 0^+ K$.*

Proof. Take any $y \in K$ and $t > 0$. We have $y_k = t\dfrac{x_k}{\| x^k \|} + \left(1 - \dfrac{t}{\| x^k \|}\right) y \in K$ and it is clear that y^k converge weakly to $t\overline{v} + y \in K$. Hence $\overline{v} \in 0^+ K$. □

Lemma 3.2 (cf. [9]). *Consider the problem* (sAVI(T,c,K)) *where K is a nonempty compact and convex subset in $\mathcal{H}$. Then,* (sAVI(T,c,K)) *has a solution.*

The following theorem is a special case of the variational inequality. However, for the sake of completeness, we provide the complete proof here.

Theorem 3.1. *Consider the problem* (sAVI(T,c,K)). *Suppose that the variational inequality problem* (sAVI(T,c,K)) *has no exceptional family. Then* (sAVI(T,c,K)) *has at least one solution.*

Proof. Suppose that, contrary to our claim, the problem (sAVI(T,c,K)) has no solution. We show that (sAVI(T,c,K)) has an exceptional family.

Taking any point, denoted as $\overline{x}$, belonging to the set K. Since (sAVI(T,c,K)) has no solution, by Lemma 2.1, there exists no $r > 0$ such that the problem $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$ has a solution $\widehat{x_r} \in K_r$ with

$$\| \widehat{x_r} - \overline{x} \| < r.$$

It is evident that K_r is a nonempty, compact and convex set for each r. Consequently, according to Lemma 3.2, we deduce that the problem $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$ has at least one solution. Therefore, there exists a sequence $\{x_r\}$ with the following property: For each r, $x_r \in \mathrm{Sol}(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$ and

$$\| x_r - \overline{x} \| = r.$$

We next claim $\{x_r\}$ is an exceptional family for (sAVI(T,c,K)). It is easy to check that $\| x_r \| \to \infty$ as $r \to \infty$. Since $x_r \in \mathrm{Sol}(\mathrm{VI}(\mathrm{T}, \mathrm{c}, \mathrm{K_r}))$, by Lemma 2.2, there exists a scalar $\mu_r \geqslant 0$ such that

$$-Tx_r - c - \mu_r (x_r - \overline{x}) \in N_k (x_r). \tag{4}$$

It follows from (4) that

$$-Tx_r - c \in N_K(x_r)$$

if $\mu_r = 0$. Thus we deduce from Lemma 2.3 that $x_r \in \mathrm{Sol(sAVI(T,c,K))}$. This is in contradiction with our assumption at the beginning of the proof. Consequently, $\mu_r > 0$. So we have shown that the sequence $\{x_r\}$ satisfies $\| x_r \| \to \infty$ as $r \to \infty$, and for each $r > 0$, there exists a real number $\mu_r > 0$ such that (4) holds. By Definition 2.2, $\{x_r\}$ is an exceptional family for (sAVI(T,c,K)). This contradicts our assumption that the problem (sAVI(T,c,K)) has no exceptional family. Hence, (sAVI(T,c,K)) has at least one solution. □

The following theorem gives a sufficient condition for the solution existence of (sAVI(T,c,K)).

Theorem 3.2. *Consider the problem* (sAVI(T,c,K)). *Suppose that*

(i) T *is positive semidefinite plus on* 0^+K*;*

(ii) *if* $\{x^k\} \subset K$ *and* $\| x^k \| \to \infty$ *as* $k \to \infty$ *then* $\limsup_{k\to\infty} \frac{\langle Tx^k, x^k \rangle}{\| x^k \|} \geq 0;$

(iii) *there exists* $\bar{x} \in K$ *such that* $\langle T\bar{x} + c, v \rangle > 0$ *for all* $v \in (0^+K) \setminus \{0\}$.

Then, the set Sol(sAVI(T,c,K)) *is nonempty*.

Proof. Take any point $\bar{x}$ in K. For any number r, we denote $\overline{\mathcal{B}} = \{x \in \mathcal{H} \mid \| x - \bar{x} \| \leqslant r\}$ and $K_r = K \cap \overline{\mathcal{B}}$.

To prove the theorem, we first show that the problem (sAVI(T,c,K)) has no exceptional family. Suppose on the contrary that (sAVI(T,c,K)) has exceptional family. By Definition 2.2, we have that $\| x_r \| \to \infty$ as $r \to \infty$, and that, for each r, there a scalar $\mu_r > 0$ such that

$$-Tx_r - c - \mu_r(x_r - \bar{x}) \in N_k(x_r). \tag{5}$$

Since (5) is satisfied with $\mu_r > 0$, it follows from Lemma 2.2 that for each $r > 0$, x_r is a solution to the variational inequality problem $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K}_\mathrm{r}))$. As the solution set to the problem $(\mathrm{sAVI}(\mathrm{T}, \mathrm{c}, \mathrm{K}_\mathrm{r}))$ is nonempty for each $r > 0$, it follows that

$$\langle Tx_r + c, \bar{x} - x_r \rangle \geq 0 \quad \forall r > 0. \tag{6}$$

Put $v^k = \| x_r \|^{-1} x_r$. One has $\| v^k \| = 1$, then there exists a subsequence of v_k weakly converges to v. Without loss of generality we can assume that $\| x_r \|^{-1} x_r$ weakly converges to some $v \in \mathcal{H}$ as $r \to \infty$. It follows from Lemma 3.1 that

$$v \in 0^+K.$$

From (6) it follows that

$$\langle Tx_r, x_r \rangle \leq \langle Tx_r + c, \bar{x} \rangle - \langle c, x_r \rangle \ \ \forall r > 0. \tag{7}$$

Multiplying both sides of (7) by $\| x_r \|^{-2}$ and letting $r \to \infty$ we obtain

$$\lim_{r\to\infty}\left\langle T\frac{x_r}{\|x_r\|},\frac{x_r}{\|x_r\|}\right\rangle \le \lim_{r\to\infty}\left(\left\langle T\frac{x_r}{\|x_r\|},\frac{\overline{x}}{\|x_r\|}\right\rangle+\left\langle c,\frac{\overline{x}}{\|x_r\|^2}\right\rangle-\left\langle c,\frac{x_r}{\|x_r\|^2}\right\rangle\right)$$
$$=\lim_{r\to\infty}\left(\left\langle T\frac{x_r}{\|x_r\|},\frac{\overline{x}}{\|x_r\|}\right\rangle+\left\langle c,\frac{\overline{x}}{\|x_r\|^2}\right\rangle-\left\langle c,\frac{x_r}{\|x_r\|^2}\right\rangle\right)=0. \tag{8}$$

Since T is a continuous linear operator, we have

$$\lim_{r\to\infty}\left\langle T\frac{x_r}{\|x_r\|},\frac{x_r}{\|x_r\|}\right\rangle=\langle Tv,v\rangle. \tag{9}$$

From (8), (9) and the positive semidefiniteness of T on 0^+K, it follows that

$$0\le\langle Tv,v\rangle=\lim_{r\to\infty}\left\langle T\frac{x_r}{\|x_r\|},\frac{x_r}{\|x_r\|}\right\rangle\le 0.$$

Hence

$$\lim_{r\to\infty}\left\langle T\frac{x_r}{\|x_r\|},\frac{x_r}{\|x_r\|}\right\rangle=0=\langle Tv,v\rangle. \tag{10}$$

Since $\langle Tv,v\rangle=0$ for all $v\in 0^+K$, and by assumption (i), it follows that

$$(T+T^*)v=0$$

or equivalent

$$Tv=-T^*v. \tag{11}$$

By dividing both sides of the (7) by $\|x^j\|$ and letting $j\to\infty$, and use assumption (ii), we get

$$\langle Tv,\overline{x}\rangle-\langle c,v\rangle\ge 0. \tag{12}$$

Combining (11) and (12) we obtain $\langle T\overline{x}+c,v\rangle\le 0$. Since $\|v^k\|=1$ and v^k converges to some v, it follows that $v\neq 0$. Thus we have shown that there exists a nonzero v such that $\langle T\overline{x}+c,v\rangle\le 0$. This, however, contradicts assumption (iii). Therefore, (sAVI(T,c,K)) has no exceptional family elements.

Since the problem (sAVI(T,c,K)) has no exceptional family elements, it follows from Theorem 3.1 that it has a solution. Thus, the proof is complete.

□

Let us mention a consequence of the theorem.

Corollary 3.1. *Consider the problem* (sAVI(T,c,K)). *Suppose that*

(i) *the operator* T *is positive semidefinite on* $\mathcal{H}$*;*

(ii) *there exists* $\overline{x}\in K$ *such that* $\langle T\overline{x}+c,v\rangle>0$ *for all* $v\in(0^+K)\setminus\{0\}$.

Then, the set Sol(sAVI(T,c,K)) *is nonempty.*

Proof. To prove the corollary, by Theorem 3.2, it suffices to verify that T is copositive plus on 0^+K. Indeed, since T is positive semidefinite on $\mathcal{H}$, $\langle Tv, v\rangle \geq 0$ for all $v \in 0^+K \subset \mathcal{H}$. Moreover, if $\langle Tv, v\rangle = 0$ then v is minimum point of the convex quadratic program $\min_{x\in\mathcal{H}}\langle (T+T^*)x, x\rangle$. By the Fermat rule, $(T+T^*)v = 0$. The proof is complete. □

In the remainder of this section, we investigate the existence of solutions to problem (sAVI(T,c,K)) where the set K is defined by

$$K = \{x \in \mathcal{H} \mid g_i(x) = \frac{1}{2}\langle x, Tx\rangle + \langle c_i, x\rangle + \alpha_i,\ i = 1, 2, \ldots, m\}, \tag{13}$$

where T_i are positive semidefinite continuous linear self-adjoint operators on $\mathcal{H}$, $c_i \in \mathcal{H}$ and α_i are real numbers.

Theorem 3.3. *Consider the problem* (sAVI(T,c,K)), *where* T *is a positive semidefinite continuous linear self-adjoint operator on* $\mathcal{H}$. *Suppose that set* K *is defined as in* (13). *Then, the set* Sol(sAVI(T,c,K)) *is nonempty if any one of the following conditions is satisfied*

(b_1) $c = 0$,

(b_2) $\big(v \in (0^+K)\setminus\{0\}, Tv = 0\big) \Rightarrow \langle c, v\rangle > 0,$

(b_3) $\big(v \in (0^+K), Tv = 0\big) \Rightarrow \big(\langle c, v\rangle \geqslant 0, \langle c_i, v\rangle = 0, \forall i \in I_1\big)$

where $I = \{1, 2, \ldots, m\}, I_1 = \{i \in I \mid T_i \neq 0\}$.

Proof. Let $f(x) := \frac{1}{2}\langle x, Tx\rangle + \langle c, x\rangle$. Since T is a positive semidefinite continuous linear self-adjoint operator on $\mathcal{H}$, by applying Theorem 3.5 in [20], we derive the convex quadratic programming problem

$$\min_{x\in K} f(x) \tag{14}$$

has a solution if one of the conditions (b_1), (b_2), (b_3) is satisfied. That is, there exists x^* in K such that $f(x) - f(x^*) \geq 0$ for all $x \in K$.

It remains to prove that x^* is a solution of (sAVI(T,c,K)). Indeed, since x^* is a solution of (14), by [4, Theorem 3.1] it follows that

$$\langle Tx^* + c, x - x^*\rangle \geq 0, \forall x \in K.$$

The proof is complete. □

4. Examples

The following example illustrates that in Theorem 3.2, one cannot omit assumption (i) while keeping the other assumptions.

Example 4.1. Consider the problem (sAVI(T,c,K)) where $\mathcal{H}=\mathbb{R}^2$, $T:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $Tx=\left(-x_2,0\right)$, $c=(\frac{1}{2},1)$ and the set K is defined

$$K=\{x=(x_1,x_2)\in\mathbb{R}^2 \mid x_1\le 0,-x_2\le 0\}.$$

We have

$$0^+K=\{v=(v_1,v_2)\in\mathbb{R}^2 \mid v_1\le 0, v_2\ge 0\}.$$

Since

$$\langle Tx,x\rangle=-x_1x_2\ge 0\,\forall x\in K,$$

the assumption (ii) in Theorem 3.2 is satisfied. For $\hat{x}=(0,1)\in K$. we have

$$\langle T\hat{x}+c,v\rangle=-\frac{1}{2}v_1+v_2>0\ \forall v\in 0^+K\setminus\{0\}.$$

Hence the assumption (iii) in Theorem 3.2 is satisfied. As

$$\langle Tv,v\rangle\ge 0\,\forall v\in 0^+K,$$

T is positive semidefinite on 0^+K . However,

$$(T+T^*)v=(-v_1,-v_2)\ne(0,0)\,\forall v\in 0^+K\setminus\{0\}.$$

In particular, for $v=(0,1)\in 0^+K$ satisfying $\langle Tv,v\rangle=0$, one does not have $(T+T^*)v=0$. Thus the assumption (i) in Theorem 3.2 is violated.

We claim that $\mathrm{Sol(sAVI(T,c,K))}=\varnothing$. Indeed, suppose that $\bar{x}=(\bar{x}_1,\bar{x}_2)$ is a solution of $(\mathrm{sAVI(T,c,K)})$. Then, by Theorem 5.3 in [11], $\bar{x}=(\bar{x}_1,\bar{x}_2)$ is a solution of $(\mathrm{sAVI(T,c,K)})$ if and only if there exists $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2$ such that

$$\begin{cases} -\bar{x}_2+\frac{1}{2}+\lambda_1=0, 1-\lambda_2=0\\ \lambda_1\ge 0,\ \lambda_2\ge 0,\ \lambda_1\bar{x}_1=0,\ \lambda_2(-\bar{x}_2)=0,\\ \bar{x}_1\le 0,\ -\bar{x}_2\le 0.\end{cases}$$

This system implies that

$$\frac{1}{2}+\lambda_1=0,\ \lambda_2=1,\ \lambda_1\ge 0,\ \bar{x}_2=0,\ \ \bar{x}_1\le 0,$$

which is impossible. Thus $\mathrm{Sol(sAVI(T,c,K))}=\varnothing$.

The following example illustrates that if condition (ii) in Theorem 3.2 is disregarded, the $(\mathrm{sAVI(T,c,K)})$ problem may not have a solution.

Example 4.2. Consider the problem $(\mathrm{sAVI(T,c,K)})$ where $\mathcal{H}=\mathbb{R}^2$, $T:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $Tx=\left(-x_1,0\right)$, $c=(2,\frac{1}{2})$ and the set K is defined by

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid g_i(x) = x_1 \le 0, g_2(x) = x_1^2 - x_2 \le 0\}.$$

We have

$$0^+ K = \{v = (v_1, v_2) \in \mathbb{R}^2 \mid v_1 = 0, v_2 \ge 0\},$$

$$\langle Tv, v\rangle = -v_1^2 = 0 \text{ and } (T + T^*)v = (-2v_1, 0) = (0,0) \quad \forall v \in 0^+ K.$$

Hence the assumption (i) in Theorem 3.2 is satisfied. For $\hat{x} = (-1,1) \in K$, we have

$$\langle T\hat{x} + c, v\rangle = \frac{1}{2} v_2 > 0 \ \ \forall v \in 0^+ K \setminus \{0\}.$$

Thus the assumption (iii) in Theorem 3.2 is satisfied. Sine $\langle Tx, x\rangle = -x_1^2 \le 0 \ \ \forall x \in K$, the assumption (ii) is violated.

We now show that $\mathrm{Sol(sAVI(T,c,K))} = \varnothing$. Indeed, suppose that $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of $\mathrm{(sAVI(T,c,K))}$. Take, for instance, $\bar{h} = x^0 - \bar{x}$ where $x^0 = (-1,2)$. It is easy to check that $\langle \nabla g_i(\bar{x}), \bar{h}\rangle < 0 \ \ i \in I(\bar{x})$ ($\nabla g_i(\bar{x})$ denotes the gradient of g_i at $\bar{x}$, $I(\bar{x}) = \{i \in \{1,2\} \mid g_i(x) = 0\}$). Hence *the Mangasarian–Fromovitz Constraint Qualification* holds at $\bar{x}$. By Proposition 1.3.4 in [7], $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of $\mathrm{(sAVI(T,c,K))}$ if and only if there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that.

$$\begin{cases} -\bar{x}_1 + 2 + \lambda_1 + 2\lambda_2 \bar{x}_1 = 0, \dfrac{1}{2} - \lambda_2 = 0 \\ \lambda_1 \ge 0, \ \lambda_2 \ge 0, \ \lambda_1 \bar{x}_1 = 0, \ \lambda_2(\bar{x}_1^2 - \bar{x}_2) = 0, \\ \bar{x}_1 \le 0, \ \bar{x}_1^2 - \bar{x}_2 \le 0. \end{cases}$$

This system implies that

$$2 + \lambda_1 = 0, \lambda_2 = \frac{1}{2}, \lambda_1 \ge 0, \bar{x}_1 \le 0, \ \bar{x}_1^2 = \bar{x}_1,$$

which is impossible. Thus $\mathrm{Sol(sAVI(T,c,K))} = \varnothing$.

The following example shows that condition (iii) in Theorem 3.2 cannot be omitted.

Example 4. 3. Consider the problem $\mathrm{(sAVI(T,c,K))}$ where $\mathcal{H} = \mathbb{R}^2$, $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $Tx = (x_1, 0)$, $c = (1,0)$ and the set K as in Example 4.2. We have

$$0^+ K = \{v = (v_1, v_2) \in \mathbb{R}^2 \mid v_1 = 0, v_2 \ge 0\}.$$

Since $\langle Tx, x\rangle \ge 0$ for all $x \in \mathbb{R}^2$ and $\mathcal{H} = \mathbb{R}^2$ is of finite dimension, it follows that assumptions (i) and (ii) of the Theorem 3.2 hold. For any $\hat{x} = (\hat{x}_1, \hat{x}_2) \in K$, we have

$$\langle T\hat{x} + c, v\rangle = 0 \ \ \forall v \in 0^+ K \setminus \{0\}.$$

Hence, the assumption (iii) of Theorem 3.2 is violated.

We now show that $\mathrm{Sol(sAVI(T, c, K))} = \varnothing$. Indeed, suppose that $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of $\mathrm{(sAVI(T, c, K))}$. Then, by Proposition 1.3.4 in [7], $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of $\mathrm{(sAVI(T, c, K))}$ if and only if there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that

$$\begin{cases} \bar{x}_1 + 1 + \lambda_1 + \lambda_2 2\bar{x}_1 = 0, -\lambda_2 = 0, \\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1\bar{x}_1 = 0,\ \lambda_2(\bar{x}_1^2 - \bar{x}_2) = 0, \\ \bar{x}_1 \leq 0,\ \bar{x}_1^2 - \bar{x}_2 \leq 0. \end{cases}$$

which is impossible. Thus $\mathrm{Sol(sAVI(T, c, K))} = \varnothing$.

Remark 4.1. Based on Example 4.3, we can deduce that the condition $\langle T\bar{x} + c, v\rangle > 0$ in assumption (ii) of Theorem 3.2 cannot be replaced by $\langle T\bar{x} + c, v\rangle \geq 0$. This implies that assumption (ii) of Theorem 3.2 cannot be weakened.

5. Conclusions

In this work, we address the semi-affine variational inequality problem in finite-dimensional Hilbert space and propose conditions for the existence of solutions to the problem. Our results are established without requiring the monotonicity of the operator or the compactness of the constraint set.

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