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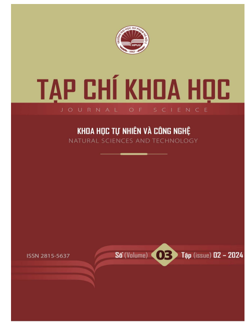

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Right-angled Artin groups and representation liftings

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Abstract

The lifting problems are interesting problems of number theory. There are many mathematicians who study lifting problems with different classes of groups. They prove the lifting problems with different classes of groups using various methods. Recently, right-angled Artin groups have attracted much attention in number theory. They have nice structure and properties. Currently, we study right-angled Artin groups with different problems related to them. One of those problems is that we want to prove the lifting problem is associated with this class of groups. We have obtained a result for this problem. In this paper, we will show that a mod p Heisenberg representations of a right-angled Artin group can be lifted to a mod p^2 representation.

Keywords: Right angled Artin groups, Heisenberg groups, liftings, Galois groups, infinite groups

1. Introduction

Let $\boldsymbol{p}$ be a fixed prime number. Let $\boldsymbol{K}$ be a field and let $\boldsymbol{G_K}$ be the absolute Galois group of $\boldsymbol{K}$. Let κ be a finite field of characteristic $\boldsymbol{p}$. In [1], the author has shown that for any field $\boldsymbol{K}$, every continuous representation $\boldsymbol{\alpha}: \boldsymbol{G_K} \to \boldsymbol{GL_2}(\boldsymbol{\kappa})$, lifts to $\boldsymbol{GL_2}(\boldsymbol{W_2}(\boldsymbol{\kappa}))$, here $\boldsymbol{W_2}(\boldsymbol{\kappa})$ is the ring of Witt vectors of length 2 over κ. This result is also written in the Proposition 3.3 in [2], see also the Theorem 6.1 in [3]. The above lifting mod $\boldsymbol{p^2}$ result for 2-dimensional mod $\boldsymbol{p}$ representations leads naturally to the study of the lifting problem for higher dimensional representations. In [2], the authors have studied the lifting problem mostly for 3-dimensional mod $\boldsymbol{p}$ representations to mod $\boldsymbol{p^2}$ representations for finite groups, absolute Galois groups of abstract fields and absolute Galois groups of local and global fields. Many mathematicians have proven lifting problems using different methods.We also have studied the methods of authors in [3]–[8] to find ways to prove our problem. In this short note, we study the lifting problem for a class of (infinite) groups, the so-called right-angled Artin groups. The

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right-angled Artin groups with their properties are studied by many mathematics as in [9]–[18]. Recall that a simple graph is a graph with no loops and no multiple edges [19]. For a finite simplicial graph $\boldsymbol{G} = (\boldsymbol{A}, \boldsymbol{E})$ with vertex set $\boldsymbol{A}$ and edge set $\boldsymbol{E}$, one can associate with it a right-angled Artin group (RAAG) $\boldsymbol{G_\Gamma}$, with a generator $\boldsymbol{u}$ for each vertex $\boldsymbol{u} \in \boldsymbol{A}$ and with a commutator relation $\boldsymbol{uv} = \boldsymbol{vu}$ for each edge $\{\boldsymbol{u}, \boldsymbol{v}\} \in \boldsymbol{E}$. For example, if the edge set $\boldsymbol{E}$ is empty then $\boldsymbol{G_\Gamma}$ is free on a set of generators $\boldsymbol{A}$. Our main result is the following theorem. (Here for a (unital) commutative ring $\boldsymbol{R}$, $\mho_{\boldsymbol{3}}(\boldsymbol{R})$ is the group of all upper triangular unipotent $\boldsymbol{n} \times \boldsymbol{n}$-matrices with entries in $\boldsymbol{R}$.)

Theorem 1.1. Let G_Γ be a right-angled Artin group and $\rho: G_\Gamma \to \mho_3(\mathbb{Z}/p\mathbb{Z})$ a group homomorphism. Then ρ lifts to a group homomorphism $\tilde{\rho}: G_\Gamma \to \mho_3(\mathbb{Z}/p^2\mathbb{Z})$.

2. Proof of the main result

Lemma 2.1. Let X and Y be the two matrices in $\mho_{\boldsymbol{3}}(\mathbb{Z}/p\mathbb{Z})$**.** If X and Y do not commute then $\mho_{\boldsymbol{3}}(\mathbb{Z}/p\mathbb{Z})$ is generated by X and Y.

Proof. Set $G = \mho_{\boldsymbol{3}}(\mathbb{Z}/p\mathbb{Z})$ and let Z be the center of G. It is well known that

$Z = \left\{\begin{bmatrix}1 & 0 & b\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \mid b \in \mathbb{Z}/p\mathbb{Z}\right\}$, which is the Flattini subgroup of G, and $G/Z \simeq \mathbb{F}_p \times \mathbb{F}_p$.

Under the identification $G/Z = \mathbb{F}_p \times \mathbb{F}_p$, the natural surjection $G \to G/Z$ becomes the homomorphism $\varphi: G \to G/Z = \mathbb{F}_p \times \mathbb{F}_p$, which is given by

$$\varphi\left(\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\right) = (a, c).$$

We write $X = \begin{bmatrix}1 & a_1 & b_1\\0 & 1 & c_1\\0 & 0 & 1\end{bmatrix}$, and $Y = \begin{bmatrix}1 & a_2 & b_2\\0 & 1 & c_2\\0 & 0 & 1\end{bmatrix}$, where $a_i,\ b_i,\ c_i$ are in $\mathbb{Z}/p\mathbb{Z}$ $(i = 1, 2)$.

Since $XY \neq YX, a_1c_2 \neq a_2c_1$. Hence $\varphi(X)$ and $\varphi(Y)$ generate $G/Z = \mathbb{F}_p \times \mathbb{F}_p$. By Burnside Basis Theorem (Theorem 4.10 in [20]), X and Y generate G.

Lemma 2.2. Let $A_i, i = 1, \dots, n$, be matrices in $\mho_{\boldsymbol{3}}(\mathbb{Z}/p\mathbb{Z})$ such that $A_iA_j = A_jA_i$, for every $i \neq j$. Then there are matrices $\widetilde{A_\iota} \in \mho_{\boldsymbol{3}}(\mathbb{Z}/p^2\mathbb{Z})$ such that $\widetilde{A_\iota}$ reduces to A_i modulo p, and $\widetilde{A_\iota}\widetilde{A_J} = \widetilde{A_J}\widetilde{A_\iota}$, for every $i \neq j$.

Proof. For each $i = 1, \dots, n$, write $A_i = \begin{bmatrix}1 & a_i & b_i\\0 & 1 & c_i\\0 & 0 & 1\end{bmatrix}$, where $a_i,\ b_i,\ c_i$ are in $\mathbb{Z}/p\mathbb{Z}$. From the condition $A_iA_j = A_jA_i$, we see that $a_ic_j = a_jc_i$, for every $i \neq j$. We consider two cases.

Case 1: There exists i such that $(a_i, c_i) \neq (0,0)$. For simplicity, we may assume that $(a_1, c_1) \neq (0,0)$. For each $i = 2, \dots, n$ from $a_1c_i = a_ic_1$, we see that $(a_i, c_i) = k_i(a_1, c_1)$ for some $k_i \in \mathbb{Z}/p\mathbb{Z}$. We also let $k_1 = 1$. Let $\widetilde{A_\iota} = \begin{bmatrix}1 & \widetilde{k_\iota}\widetilde{a_1} & \widetilde{b_\iota}\\0 & 1 & \widetilde{k_\iota}\widetilde{c_1}\\0 & 0 & 1\end{bmatrix}$, where $\widetilde{k_\iota}$ (respectively $\widetilde{a_1},\ \widetilde{b_\iota},\ \widetilde{c_1}$) is an element in $\mathbb{Z}/p^2\mathbb{Z}$ which reduces modulo p to k_i (respectively $a_1,\ b_i,\ c_1$). Then each $\widetilde{A_\iota}$ reduces to A_i modulo p and $\widetilde{A_\iota}\widetilde{A_J} = \widetilde{A_J}\widetilde{A_\iota}$, for every $i \neq j$.

Case 2: $(a_i, c_i) = (0,0)$ for every i. In this case, let $\widetilde{A_i} = \begin{bmatrix} 1 & 0 & \widetilde{b_i} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\widetilde{b_i}$ is any element in $\mathbb{Z}/p^2\mathbb{Z}$ that reduces modulo p to b_i. Then each $\widetilde{A_i}$ reduces to A_i modulo p and $\widetilde{A_i}\widetilde{A_j} = \widetilde{A_j}\widetilde{A_i}$, for every $i \neq j$.

We immediately obtain the following corollary.

Corollary 2.3. Theorem 1.1 holds if Γ is complete.

Proof of Theorem 1.1. We proceed by induction on the number of vertices of the graph Γ. Suppose that Γ is not connected. Then $\Gamma = \Gamma_1 \sqcup \Gamma_2$ is the disjoint union of two subgraphs. Then $G_\Gamma = G_{\Gamma_1} * G_{\Gamma_2}$ is the free product of G_{Γ_1} and G_{Γ_2}. The statement follows from the induction hypothesis.

So we assume that $\Gamma = (V, E)$ is not connected, $V = \{v_1, \dots, v_n\}$, and n is fixed. Let $A_i = \rho(v_i)$. Now we proceed by backward induction on the number of edges E. If Γ is complete then the statement from Corollary 2.3. Suppose that Γ is not complete. Then there are two vertices v, u such that $\{v, u\}$ is not an edge of the graph. Let $v = u_0, u_1, u_2, \dots, u_r = u$ be a shortest path that connects v and u. Then $r \geq 2$. By reindexing if necessary, we may and shall assume that $v = v_1, u_1 = v_2$ and $u_2 = v_3$. Then $\{v_1, v_3\}$, is not an edge.

If $A_1A_3 = A_3A_1$, we replace Γ by $\Gamma' = (V', E')$, where $V' = V$ and $E' = E \sqcup \{\{v_1, v_3\}\}$. Then by induction hypothesis applied to (V, E'), there are matrices $\widetilde{A_i} \in \mho_3(\mathbb{Z}/p^2\mathbb{Z})$ such that each $\widetilde{A_i}$ reduces to A_i modulo p and $\widetilde{A_i}\widetilde{A_j} = \widetilde{A_j}\widetilde{A_i}$ for every $1 \leq i \leq j \leq n$ with $\{i, j\} \in E'$. These relations imply that we can define a homomorphism $\tilde{\rho}: G_\Gamma \to \mho_3(\mathbb{Z}/p^2\mathbb{Z})$ by $\rho(v_i) = \widetilde{A_i}, \forall\, i = 1, \dots, n$. Clearly, $\tilde{\rho}$ is a lift of ρ.

If $A_1A_3 \neq A_3A_1$ then by Lemma 2.1, $\mho_3(\mathbb{Z}/p\mathbb{Z})$ is generated by A_1 and A_3. Hence A_2 is the center of $\mho_3(\mathbb{Z}/p\mathbb{Z})$ and A_2 is of the form $A_2 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, for some $a \in \mathbb{Z}/p\mathbb{Z}$. By the induction hypothesis applying the graph $\Gamma_2 - \{v_2\}$, there are matrices $\widetilde{A_i} \in \mho_3(\mathbb{Z}/p^2\mathbb{Z})$, $2 \leq i \leq n$, such that each $\widetilde{A_i}$ reduces to A_i modulo p and $\widetilde{A_i}\widetilde{A_j} = \widetilde{A_j}\widetilde{A_i}$ for every $2 \leq i \leq j \leq n$ with $\{i, j\} \in E$.

We pick any element $\tilde{a}$ that reduces to a modulo p. Set $\widetilde{A_2} = \begin{bmatrix} 1 & 0 & \tilde{a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $\widetilde{A_2}\widetilde{A_i} = \widetilde{A_i}\widetilde{A_2}$, for every i, and hence $\widetilde{A_i}\widetilde{A_j} = \widetilde{A_j}\widetilde{A_i}$ for every $1 \leq i \leq j \leq n$ with $\{i, j\} \in E$. These relations imply that we can define a homomorphism $\tilde{\rho}: G_\Gamma \to \mho_3(\mathbb{Z}/p^2\mathbb{Z})$ by $\rho(v_i) = \widetilde{A_i}, \forall\, i = 1, \dots, n$. Clearly, $\tilde{\rho}$ is a lift of ρ.

3. Conclusions

We prove that a mod $\boldsymbol{p}$ Heisenberg representations of a right angled Artin group can be lifted to a mod $\boldsymbol{p}^2$ representation

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