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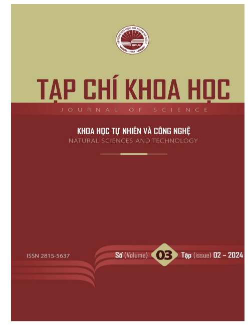

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A non-existence result for higher-order Hardy-Hénon inequality on punctured balls

Thi-Ngoan Tran[a,b], Van-Tuan Tran[b*]

[a]*Thai Binh University, Thai Binh, Vietnam*
[b]*Hanoi Pedagogical University 2, Vinh Phuc, Vietnam*

Abstract

Let n and m be two positive integers such that $n > 2m \geq 4$. Let σ and p be real such that $\sigma < -2m$ and $p > 1$. In this note, we are mainly concerned with non-negative and classical solutions of the high-order harmonic inequality

$$(-\Delta)^m u \geq |x|^\sigma u^p,$$

on the punctured ball $\boldsymbol{B_R} \setminus \{\boldsymbol{0}\} \subset \mathbf{R}^{\boldsymbol{n}}$. Using the method of test functions, the Hölder's inequality, and integral estimates, we will prove that this inequality has no $\boldsymbol{C^{2m}}$ positive solution satisfying some sufficient conditions. It should be mentioned that our result, see Theorem 1.1 in the next section, in the high-order setting is analogous to that of Laptev for the case $\boldsymbol{m = 2}$.

Keywords: Hardy–Hénon polyharmonic equation, non–negative and classical solution, existence and non–existence, weak and strong super–polyharmonic properties

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1. Introduction

Let n and m be two integers such that $n > 2m \geq 4$. Let σ and p be real such that $\sigma < -2m$ and $p > 1$. In $\mathbf{R}^n$, we use the notation $|x|$ as the standard norm of a vector $x = (x_1, \dots, x_n)$ in $\mathbf{R}^n$, i.e. $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and B_R is the ball of radius R, centered at the origin. We also write Δ as the Laplace operator: $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.

We are concerned with non-negative, non-trivial and classical solutions to the following functional inequality involving the high-order Laplace operator:

$$(-\Delta)^m u(x) \geq |x|^\sigma u^p(x) \ \text{ in } B_R \setminus \{0\} \subset \mathbf{R}^n. \tag{1}$$

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* Corresponding author, E-mail: tranvantuan@hpu2.edu.vn

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The reason why we are interested in solutions to (1) goes back to a striking result due to Laptev, see Theorem 2 in [1]. This result says that the inequality (1) holds for $m = 2$, i.e. if $n \ge 5, p > 1$, and $\sigma \le -4$, the functional inequality

$$(-\Delta)^2 u(x) \ge |x|^\sigma u^p(x) \text{ in } B_R \setminus \{0\} \subset \mathbf{R}^n,$$

admits no punctured solution u satisfying

$$\int_{\partial B_R} \Delta u d\sigma \le 0. \tag{2}$$

See also Section 6 in [2]. To obtain the above result, Laptev used the method of test functions, which depends heavily on the conditions $n \ge 5$ and $\sigma \le -4$.

Of interest in this paper is to extend Laptev's result to a higher order setting. The main result is as follows.

Theorem 1.1. Let $n > 2m$ and $R > 0$. Then the problem

$$(-\Delta)^m u \ge |x|^\sigma u^p \text{ in } B_R \setminus \{0\} \subset \mathbf{R}^n,$$

with $p > 1$ and $\sigma < -2m$ does not admit any non-negative, non-trivial punctured solution u satisfying:

$$\int_{\partial B_R} (-\Delta)^k u d\sigma \ge 0, \tag{3}$$

for all $1 \le k \le m - 1$.

It should be noted that without assuming the boundary conditions (3), Theorem 1.1 is in general not true. This was already discussed in [3] in the case $m = 2$. For some other related results, the reader can consult in the recent works [1], [4]–[20].

To prove Theorem 1.1, we first recall a preliminary result in Section 2. Second, the proof of Theorem 1.1 will be shown in Section 3.

2. A technical result

The following lemma is the initial step of showing Theorem 1.1.

Lemma 2.1. There exists a function $\Phi \in C^{2m}(B_1 \setminus \{0\})$ satisfying $\Phi > 0$ in $B_1 \setminus \{0\}$,

$$(-\Delta)^m \Phi = 0 \text{ in } B_1 \setminus \{0\},$$

and for all $0 \le i \le m - 1$,

$$(-\Delta)^i \Phi(x) = 0 \text{ in } B_1 \setminus \{0\} \quad \text{and} \quad \frac{\partial}{\partial \nu}(-\Delta)^i \Phi(x) \le 0 \text{ on } \partial B_1.$$

Proof. For clarity, we denote by Φ_m the desired function. Let us construct by induction on m that

$$\Phi_m(x) = \beta_m |x|^{2m-n} + \sum_{i=0}^{m-1} \beta_i |x|^{2i} \text{ in } B_1 \setminus \{0\}, \tag{4}$$

for some appropriate constant $\beta_m > 0$ and $\beta_i \in \mathbf{R}$.

For $m = 1$, the function Φ_1 is given as follows

$$\Phi_1(x) = c_1 \left(\frac{1}{|x|^{n-2}} - 1 \right),$$

where $c_1 > 0$ is a constant chosen in such a way that $c_1/|x|^{n-2}$ is the fundamental solution to the Laplace equation in $\mathbf{R}^n$ with $n > 2$. Suppose that Φ_m is given by (4), we now construct Φ_{m+1} as follows.

Let constants $\gamma_i (0 \le i \le m+1)$ be given as follows

$$\begin{cases} \gamma_{m+1} = \dfrac{\beta_m}{2m(n-2m)} > 0, \\ \gamma_i = \dfrac{\beta_i}{2i(2i+n-2)} \text{ for } 0 \le i \le m, \\ \gamma_0 = -\displaystyle\sum_{i=1}^{m+1} \gamma_i. \end{cases}$$

We define

$$\Phi_{m+1}(x) = \gamma_{m+1}|x|^{2m+2-n} + \sum_{i=0}^{m} \gamma_i |x|^{2i}.$$

For the positivity of Φ_{m+1}, it can be seen that for any $x \in B_1 \setminus \{0\}$,

$$\Phi_{m+1}(x) > \gamma_{m+1} + \gamma_0 + \sum_{i=1}^{m} \max(\gamma_i, 0) \ge 0.$$

We now prove that $(-\Delta)^{m+1}\Phi_{m+1} = 0$ in $B_1 \setminus \{0\}$. Indeed, using the following computations for any a,

$$\nabla|x|^a = ax|x|^{a-2}, \quad \Delta|x|^a = a(a+n-2)|x|^{a-2}, \tag{5}$$

we obtain

$$\begin{aligned} (-\Delta)\Phi_{m+1} &= -\gamma_{m+1}(2m-n)((2m+2-n)+n-2)|x|^{2m-n} \\ &\quad -\sum_{i=1}^{m} \gamma_i(2i)(2i+n-2)|x|^{2i-2} \\ &= 2m(n-2m)\gamma_{m+1}|x|^{2m-n} - 2\textstyle\sum_{i=1}^{m} i(2i+n-2)\gamma_i|x|^{2i}. \end{aligned}$$

It yields $(-\Delta)\Phi_{m+1} = \Phi_m$ for any $x \in B_1 \setminus \{0\}$. Hence, we get that $(-\Delta)^{m+1}\Phi_{m+1} = 0$ in $B_1 \setminus \{0\}$.

Next, let us verify the boundary conditions. For $1 \le i \le m$, we have

$$(-\Delta)^i \Phi_{m+1} = (-\Delta)^{i-1}\Phi_m = 0 \text{ on } \partial B_1,$$

and

$$\frac{\partial}{\partial \nu}(-\Delta)^i \Phi_{m+1} = \frac{\partial}{\partial \nu}(-\Delta)^{i-1}\Phi_m \le 0 \text{ on } \partial B_1.$$

We are left to prove that

$$\Phi_{m+1} = 0 \text{ and } \frac{\partial \Phi_{m+1}}{\partial \nu} \le 0 \text{ on } \partial B_1.$$

Clearly, we have that $\Phi_{m+1}|_{\partial B_1} = 0$ from the choice of γ_0. Together with (5), we obtain that

$$\frac{\partial \Phi_{m+1}}{\partial \nu}\bigg|_{\partial B_1} = \left(\frac{\Phi_{m+1}}{|x|^2}\right)\bigg|_{\partial B_1} = 0.$$

The construction of Φ_m is complete. □

3. Proof of main result

Have Lemma 2.1 at hand, we are ready to illustrate Theorem 1.1.

Let $n > 2m$. Fix some $\Phi \in C^{2m}(B_1 \setminus \{0\})$ satisfying $\Phi > 0$ in $B_1 \setminus \{0\}$,

$$(-\Delta)^m \Phi = 0 \text{ in } B_1 \setminus \{0\},$$

and

$$(-\Delta)^i \Phi(x) = 0,\ \frac{\partial}{\partial v}(-\Delta)^i \Phi(x) \le 0,\ \forall 0 \le i \le m-1, x \in \partial B_1.$$

By the construction of Φ in (4), there exists $C > 0$ such that

$$\left|\nabla^k \Phi(x)\right| \le C|x|^{2m-n-k},\ \forall x \in B_1 \setminus \{0\}, 0 \le k \le 2m.$$

Assume that $u \in C^{2m}(\mathbf{R}^n \setminus \{0\})$ is a nonnegative solution to (1) with $\sigma < -2m$. Let

$$\varphi(x) = \phi_\epsilon(x)\Phi_R(x),$$

where

$$\Phi_R(x) = \Phi\left(\frac{x}{R}\right),\ \phi_\epsilon(x) = \psi\left(\frac{x}{\epsilon}\right),$$

with a radial function $\psi \in C^\infty(\mathbf{R}^n)$ supported in $\mathbf{R}^n \setminus B_1$ and $\psi \equiv 1$ in $\mathbf{R}^n \setminus B_2$.

For any $0 < 2\epsilon < R$, taking φ as test function to the equation, applying integration by parts, we get

$$\int_{B_R} |x|^\sigma u^p \varphi \, dx \le \int_{B_R} u(-\Delta)^m \varphi \, dx + \sum_{k=0}^{m-1} \int_{\partial B_R} (-\Delta)^k u \frac{\partial}{\partial v}[(-\Delta)^{m-1-k}\varphi].$$

Thanks to (3) and $\varphi \equiv \Phi_R$ near ∂B_R, we get

$$\int_{B_R} |x|^\sigma u^p \varphi dx \le \int_{B_R} u(-\Delta)^m \varphi dx.$$

Applying the Hölder's inequality, there holds

$$\int_{B_R} |x|^\sigma u^p \varphi dx \le \int_{B_R} |x|^{-\frac{\sigma}{p-1}} \varphi^{-\frac{1}{p-1}} |\Delta^m \varphi|^{\frac{p}{p-1}} dx.$$

As $(-\Delta)^m \Phi_R = 0$ in B_R, we have, for $x \in B_1 \setminus \{0\}$,

$$\begin{aligned} |\Delta^m \varphi|(x) = |\Delta^m(\phi_\epsilon \Phi_R)|(x) &\le C \sum_{k=0}^{2m} \left|\nabla^{2m-k}\phi_\epsilon(x)\right|\left|\nabla^k \Phi_R(x)\right| \\ &\le C \sum_{k=0}^{2m} \epsilon^{k-2m} R^{-k} \left|\nabla^{2m-k}\psi\left(\frac{x}{\epsilon}\right)\right| \left|\nabla^k \Phi\left(\frac{x}{R}\right)\right| \\ &\le C R^{n-2m} \sum_{k=0}^{2m} \epsilon^{k-2m} |x|^{2m-k-n} \left|\nabla^{2m-k}\psi\left(\frac{x}{\epsilon}\right)\right|. \end{aligned}$$

Keep in mind that $2m - k - n < 0$ for $k \ge 0$. Hence for $x \in B_{2\epsilon} \setminus B_\epsilon$, there holds

$$|\Delta^m \varphi|(x) \le C R^{n-2m} \epsilon^{-n} \sum_{k=0}^{2m} \left|\nabla^{2m-k}\psi\left(\frac{x}{\epsilon}\right)\right|.$$

For any $R > 0$ fixed, we can claim that, for $\epsilon > 0$ small enough

$$\int_{B_{2\epsilon}\setminus B_\epsilon} |x|^{\frac{-\sigma}{p-1}}\varphi^{-\frac{1}{p-1}}|\Delta^m\varphi|^{\frac{p}{p-1}}dx$$
$$\leq C\epsilon^{-\frac{np}{p-1}}\sum_{k=1}^{2m}\int_{B_{2\epsilon}\setminus B_\epsilon}|x|^{\frac{-\sigma}{p-1}}\left|\nabla^k\psi\left(\frac{x}{\epsilon}\right)\right|^{\frac{p}{p-1}}\psi\left(\frac{x}{\epsilon}\right)^{-\frac{1}{p-1}}\Phi\left(\frac{x}{R}\right)^{-\frac{1}{p-1}}dx$$
$$\leq C\epsilon^{-\frac{np}{p-1}}\sum_{k=1}^{2m}\int_{B_{2\epsilon}\setminus B_\epsilon}|x|^{\frac{n-2m-\sigma}{p-1}}\left|\nabla^k\psi\left(\frac{x}{\epsilon}\right)\right|^{\frac{p}{p-1}}\psi\left(\frac{x}{\epsilon}\right)^{-\frac{1}{p-1}}dx$$
$$= C\epsilon^{-\frac{2m+\sigma}{p-1}}\sum_{k=1}^{2m}\int_{B_2\setminus B_1}|y|^{\frac{n-2m-\sigma}{p-1}}\left|\nabla^k\psi(y)\right|^{\frac{p}{p-1}}\psi(y)^{-\frac{1}{p-1}}dy =: C_0\epsilon^{-\frac{2m+\sigma}{p-1}},$$

here we used again the property of Φ near the origin, namely $\Phi(x) \lesssim |x|^{2m-n}$ for small $|x|$. Remark that the constants C depend on R but remain independent on $\epsilon > 0$ small. We can choose suitable ψ such that $C_0 < \infty$. Moreover, as $\varphi \equiv \Phi_R$ in $B_R \setminus B_{2\epsilon}$ and $\varphi \equiv 0$ in B_ϵ since ψ is supported in $\mathbf{R}^n \setminus B_1$, there holds

$$\begin{aligned}\int_{B_R}|x|^\sigma u^p\varphi dx &\leq \int_{B_R}|x|^{-\frac{\sigma}{p-1}}\varphi^{-\frac{1}{p-1}}|\Delta^m\varphi|^{\frac{p}{p-1}}dx\\ &= \left(\int_{B_R\setminus B_{2\epsilon}}+\int_{B_{2\epsilon}\setminus B_\epsilon}+\int_{B_\epsilon}\right)|x|^{-\frac{\sigma}{p-1}}\varphi^{-\frac{1}{p-1}}|\Delta^m\varphi|^{\frac{p}{p-1}}dx\\ &= \int_{B_{2\epsilon}\setminus B_\epsilon}|x|^{\frac{-\sigma}{p-1}}\varphi^{-\frac{1}{p-1}}|\Delta^m\varphi|^{\frac{p}{p-1}}dx\\ &\leq C_0\epsilon^{-\frac{2m+\sigma}{p-1}}.\end{aligned}$$

Let $\epsilon \to 0$, recall that $2m + \sigma < 0$ and $p > 1$, we conclude then $|x|^\sigma u^p \Phi_R = 0$ in $B_R \setminus \{0\}$, hence $u = 0$ in $B_R \setminus \{0\}$.

4. Conclusion

In this paper, we have investigated the non-existence of positive solutions to the functional inequality involving the Laplace operator of order m on the punctured ball in $\mathbf{R}^n$, where $n > 2m \geq 4$. Based on the test function methods, we obtain the main result, Theorem 1.1, which is a generalization of Theorem 2 in [1].

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