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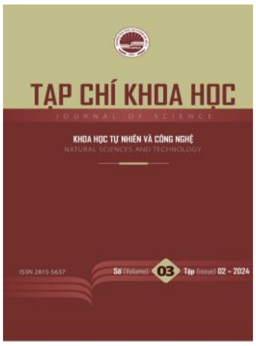

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Viscosity solutions of the augmented k-Hessian equations in exterior domains

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Abstract

This paper examines the Dirichlet problem for augmented k-Hessian equations in exterior domains. Building upon our previous results on the viscosity solutions to the Dirichlet problems for augmented k-Hessian equations in the bounded domain, and L. Dai, J. Bao's method to the k-Hessian equations in exterior domains, a sufficient condition for the existence and uniqueness of viscosity solutions to the Dirichlet problem for the augmented k-Hessian equations in exterior domains have been proven. During the process, a slight adjustment to the result on the existence and uniqueness of viscosity solutions to the problem in the bounded domains has been made for use in the present situation.

Keywords: Augmented k-Hessian equations, viscosity solutions, subsolution, (A,k)-convex function, exterior domain

1. Introduction

The viscosity solution of partial differential equations was first introduced for the first-order Hamilton-Jacobi equations in the early 1980s. This generalized solution concept has been extended to second-order nonlinear elliptic partial differential equations and has many applications, see [1]–[5] and references therein.

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Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain, $k \in \{1, 2, \ldots, n\}, \mathbb{M}^n$ the set of all $n \times n$ positive definite symmetric matrices with the norm of the matrix $X = [x_{ij}]$ given by $\|X\| = \max |x_{ij}|$. For $X \in \mathbb{M}^n$, we denote $\mu(X) = (\mu_1, \cdots, \mu_n)$ the vector of n eigenvalues of X,

$$\sigma_k(\mu) = \sigma_k(\mu_1, \cdots, \mu_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \mu_{i_1} \cdots \mu_{i_k}, \quad \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{M}^n$$

the basic symmetric polynomial of degree k. We consider the augmented k-Hessian equation

$$[\sigma_k(\mu(D^2 v(x) - A(x, v(x), Dv(x))))]^{1/k} = f(x), \quad x \in \mathcal{D}, \tag{1}$$

subject to

$$v(x) = \psi(x), \quad x \in \partial\mathcal{D}, \tag{2}$$

where $A: \overline{\mathcal{D}} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{M}^n$, $f: \mathcal{D} \to \mathbb{R}$, and $\psi: \partial\mathcal{D} \to \mathbb{R}$ are given continuous mappings, $f > 0$ in $\mathcal{D}$.

When $A \equiv 0$, Equation (1) is often called k-Hessian equation. It is well-known that the k-Hessian equation is second-order nonlinear, and is elliptic only for k-convex functions (X. J. Wang [6]). The k-Hessian equation class includes the Monge-Ampere equations (when $k = n$) and the Poisson equations (when $k = 1$). It has many important applications, especially in conformal mapping problems, and curvature theory [6]–[8].

The augmented k-Hessian equations appear when studying the optimal transport problems. When the domain $\mathcal{D} = \Omega$ is bounded, some properties of classical solutions to the Dirichlet problem (1), (2) have been studied [9], [10], and some sufficient conditions for the existence and uniqueness of viscosity solutions to that problem were proved in [11] when the data of the problem are not smooth enough. In the case $\mathcal{D} = \mathbb{R}^n \setminus \overline{\Omega}$ is an exterior domain, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and contains origin, the existence of solution with prescribed asymptotic behavior to the problem (1), (2) has studied in [7] and [8] (for $A \equiv 0, k = n$), in [12] and [13] (for $A \equiv 0, f \equiv 1$), in [14] and [15] (for $A \equiv 0$, f is unbounded and has a special growth).

In this paper, we establish a sufficient condition for the existence, uniqueness of viscosity solution with prescribed asymptotic behavior to the problem (1), (2) on the exterior $\mathcal{D} = \mathbb{R}^n \setminus \overline{\Omega}$, in the case $A(x, z, p) \le 0$ and $f(x)$ is bounded.

2. Research content

From now on, we always assume that $k \in \{1, 2, \ldots, n\}$, $\mathcal{D} = \mathbb{R}^n \setminus \overline{\Omega}$ is an exterior domain, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and contains origin 0, $A: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{M}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ are given continuous mappings, $f(x) > 0$, and

$$\Gamma_k := \{\mu \in \mathbb{R}^n : \sigma_j(\mu) > 0, \forall j = 1, 2, \cdots, k\}.$$

It is well-known that

$$\Gamma_n = \{\mu \in \mathbb{R}^n : \mu_j > 0, \forall j = 1, 2, \cdots, n\}, \quad \Gamma_i \subset \Gamma_j, \forall i > j.$$

For convenience, we will recall the concept of (A,k)-convex function and the concept of viscosity solution to Problem (1), (2).

Definition 2.1. ([11]). Given a pair (A,k). A function $v \in C(\overline{\mathfrak{D}})$ is said to be $(A,k)-convex$ on $\mathfrak{D}$ iff for any $\varphi \in C^2(\mathfrak{D})$, φ touches v from below at $x_0 \in \mathfrak{D}$ we have

$$\mu(D^2\varphi(x_0) - A(x_0, \varphi(x_0), D\varphi(x_0))) \in \overline{\Gamma}_k.$$

Remark 2.2. It is clear that if $v \in C^2(\overline{\mathfrak{D}})$ and v is (A,k)-convex on $\mathfrak{D}$ then,

$$\mu(D^2v(x) - A(x, v(x), Dv(x))) \in \overline{\Gamma}_k, \quad \forall x \in \mathfrak{D},$$

and for C^2-functions, the $(0,n)$-convexity is exactly the usual convexity.

Definition 2.3. ([15]). A function $u \in C(\mathfrak{D})$ is called a *viscosity subsolution* to Equation (1) if for any $y \in \mathfrak{D}$, any (A,k)-convex function $\xi \in C^2(\mathfrak{D})$ satisfying

$$u(x) \le \xi(x),\ x \in \mathfrak{D}; \quad u(y) = \xi(y),$$

we have

$$[\sigma_k(\mu(D^2\xi(y) - A(y, \xi(y), D\xi(y))))]^{1/k} \ge f(y),$$

A function $u \in C(\mathfrak{D})$ is called a *viscosity supersolution* to Equation (1) if for any $y \in \mathfrak{D}$, any (A,k)-convex function $\xi \in C^2(\mathfrak{D})$ satisfying

$$u(x) \ge \xi(x),\ x \in \mathfrak{D}; \quad u(y) = \xi(y),$$

we have

$$[\sigma_k(\mu(D^2\xi(y) - A(y, \xi(y), D\xi(y))))]^{1/k} \le f(y).$$

A function $u \in C(\mathfrak{D})$ is called a *viscosity solution* to Equation (1) if u is both a viscosity subsolution and a viscosity supersolution to (1).

A function $u \in C(\overline{\mathfrak{D}})$ is called a *viscosity subsolution* (resp. *viscosity supersolution, viscosity solution)* to the problem (1), (2) if u is a viscosity subsolution (resp. viscosity supersolution, viscosity solution) to Equation (1) and $u \le$ (resp. $\ge, =)\psi$ on $\partial\mathfrak{D}$.

Remark 2.4. By [11, Theorem 2.2], every viscosity subsolution and viscosity supersolution to the equation (1) is (A,k)-convex on $\mathfrak{D}$.

Lemma 2.5. *Let $\mathfrak{D} \subset \mathbb{R}^n$ be an arbitrary domain, $f \in C(\mathbb{R}^n)$ be nonnegative. Suppose that (A,k)-convex functions $u_1 \in C(\overline{\mathfrak{D}}), u_2 \in C(\mathbb{R}^n)$ are viscosity subsolutions to the equation (1) respectively in $\mathfrak{D}$ and $\mathbb{R}^n$. Moreover,*

$$u_2 \le u_1, \quad x \in \overline{\mathfrak{D}}; \quad u_1 = u_2, \quad x \in \partial\mathfrak{D}. \tag{3}$$

Set

$$v(x)=\begin{cases}u_1(x), & x\in\mathcal{D},\\ u_2(x), & x\in\mathbb{R}^n\setminus\mathcal{D}.\end{cases}$$

Then v *is a viscosity subsolution of the equation* (1) *in* $\mathbb{R}^n$.

Proof. Given $y\in\mathbb{R}^n$, $\xi\in C^2(\mathbb{R}^n)$ be an (A,k) -convex function satisfying $v(y)=\xi(y)$,

$$v(x)\le\xi(x),\quad x\in\mathbb{R}^n. \tag{4}$$

If $y\in\mathcal{D}$, then we get

$$u_1(y)=v(y)=\xi(y),\quad u_1(x)=v(x)\le\xi(x),\quad x\in\mathcal{D}.$$

Hence,

$$\sigma_j(\mu(D^2\xi(y)-A(y,\xi(y),D\xi(y))))\ge 0,\quad 1\le j\le k,$$

$$\sigma_k(\mu(D^2\xi(y)-A(y,\xi(y),D\xi(y))))\ge f(y).$$

If $y\notin\mathcal{D}$, then we obtain

$$u_2(y)=v(y)=\xi(y),\quad u_2(x)=v(x)\le\xi(x),\quad x\in\mathcal{D}.$$

From (3), (4), $u_2(x)\le\xi(x)$ for all $x\in\mathbb{R}^n$. Therefore,

$$\sigma_j(\mu(D^2\xi(y)-A(y,\xi(y),D\xi(y))))\ge 0,\quad 1\le j\le k,$$

$$\sigma_k(\mu(D^2\xi(y)-A(y,\xi(y),D\xi(y))))\ge f(y).$$

The proof is complete.

Now we introduce some assumptions for $A(x,z,p)$ and $f(x)$.

(AF_1): For each $t>0$, there exists a locally continuous module $\eta_{A,t}$ on $[0,\infty)$ satisfies

$$A(x,z,p)-A(y,z,p)\le\eta_{A,t}(|x-y|(1+|p|))I,\quad \forall x,y\in\mathbb{R}^n, |z|\le t, p\in\mathbb{R}^n;$$

(AF_2): $D_zA(x,z,p)\ge 0; \forall(x,z,p)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$;

(AF_3): $\det(-A(x,z,p))\ge[f(x)]^k,\quad (x,z,p)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$;

(AF_4): For each $t>0$, there exists a positive constant $C_{f,t}$ and a locally continuous module $\eta_{f,t}$ such that

$$|f(x)-f(y)|\le\eta_{f,t}(|x-y|), \forall x,y\in\mathbb{R}^n, |z|\le t, p\in\mathbb{R}^n.$$

(AF_5): Let Ω be a C^2, bounded, and strictly convex domain in $\mathbb{R}^n;\psi\in C(\partial\Omega)$. The Dirichlet problem

$$\begin{cases}\Delta v(x)=\mathrm{Tr}(A(x,v(x),Dv(x)), & x\in\Omega,\\ v(x)=\psi(x), & x\in\partial\Omega\end{cases}$$

has a classical solution, where $\mathrm{Tr}A$ stands for the trace of matrix A.

Remark 2.6. Some sufficient conditions for assumption (AF_5) have been established in some documents, for instance: in [16] for $\mathrm{Tr}A(x,v,Dv) = f(x)$, in [17] for $\mathrm{Tr}A(x,v,Dv) = f(v), \psi = 0$, in [18] for $\psi = 0$, and in [19], Theorem 15.10 for the general case.

According to the proof of Theorems 2.3, 2.4 in [11], but using the assumption (AF_5) instead of using the sufficient conditions for (AF_5), we have the following result on the existence and uniqueness to the Dirichlet problem for the augmented k-Hessian equation in bounded domains:

Theorem 2.7. *Let Ω be a C^2, bounded, and strictly convex domain in $\mathbb{R}^n; \psi \in C(\partial\Omega)$; $f \in C(\Omega)$, $f(x) > 0$. Moreover, suppose that the assumptions (AF_1)-(AF_5) are satisfied. Then the following problem has a unique viscosity solution:*

$$\begin{cases} [\sigma_k(\mu(D^2u(x) - A(x,u(x),Du(x))))]^{1/k} = f(x), & x \in \Omega, \\ u(x) = \psi(x), & x \in \partial\Omega. \end{cases}$$

Now, we are ready to establish the existence and uniqueness for the considering problem in exterior domains.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n, (n \geq 3)$ be a bounded, strictly convex, C^2 domain, which contains 0, $\mathfrak{D} = \mathbb{R}^n \setminus \overline{\Omega}$; $\psi \in C^2(\partial\Omega)$; $f \in C(\mathbb{R}^n), 0 < \inf_{\mathbb{R}^n} f \leq \sup_{\mathbb{R}^n} f < \infty$. Moreover, we assume that $A(x,z,p) \leq 0$, and satisfies the assumptions (AF_1)-(AF_5). Then there exists γ_0 such that for any $\gamma > \gamma_0$, there exists a unique viscosity solution to the problem in exterior domains (1), (2) such that*

$$\limsup_{x\to\infty}\left(|x|^{n-2}\left|u(x) - \left(\frac{\gamma_*}{2}|x|^2 + \gamma\right)\right|\right) < \infty, \tag{5}$$

where $\gamma_* = \left(\dfrac{1}{C_n^k}\right)^{1/k} \sup_{\mathbb{R}^n} f$.

Proof. We first construct a viscosity subsolution v_t to the problem (1), (2). For each $t > -1$, let

$$v_t(x) = \min_{\partial\Omega}\psi - \int_{\sqrt{\gamma_*}|x|}^{\bar{r}} (r^n + t)^{1/n} dr,\ x \in \mathbb{R}^n,$$

where $\bar{r} = 2\sqrt{\gamma_*}\,\mathrm{diam}\Omega$. We have $v_t \in C(\mathbb{R}^n)$, and $v_t \leq \psi$ on $\partial\mathfrak{D}$. Set

$$\rho(t) = \min_{\partial\Omega}\psi + \int_{\bar{r}}^{\infty} r\left[\left(1 + \frac{t}{r^n}\right)^{1/n} - 1\right] dr - \frac{1}{2}\bar{r}^2.$$

Then

$$v_t(x) = \frac{\gamma_*}{2}|x|^2 + \rho(t) - \int_{\sqrt{\gamma_*}|x|}^{\infty} r\left[\left(1 + \frac{t}{r^n}\right)^{1/n} - 1\right] dr,\ x \in \mathbb{R}^n. \tag{6}$$

By direct calculation, we have

$$D_{ij}v_t(x)=\left(|z|^n+t\right)^{\frac{1}{n}-1}\gamma_*\left[\left(|z|^{n-1}+\frac{t}{|z|}\right)\delta_{ij}-\frac{tz_iz_j}{|z|^3}\right],\ |x|>0,$$

here $z=\sqrt{\gamma_*}x.$ By rotating the coordinates, we may assume that $z=(R,0,\cdots,0)^T,$ and therefore

$$D^2v_t=\left(R^n+t\right)^{\frac{1}{n}-1}\gamma_*\ \mathrm{diag}\left(R^{n-1},R^{n-1}+\frac{t}{R},\cdots,R^{n-1}+\frac{t}{R}\right),$$

where $R=|z|$. From this and the fact that $A(x,z,p)\le 0,$ we have

$$\sigma_j\left(\mu(D^2v_t(x)-A(x,v_t(x),Dv_t(x)))\right)\ge\sigma_j\left(\mu(D^2v_t(x))\right)>0,\forall j=1,\dots,k,$$

therefore,

$$\mu(D^2v_t(x)-A(x,v_t(x),Dv_t(x)))\in\Gamma_k,\quad\forall|x|>0.$$

By Newton-Maclaurin inequality ([19], p. 7),

$$\begin{aligned}\left[\sigma_k\left(\mu(D^2v_t-A(x,v_t,Dv_t))\right)\right]^{1/k}&\ge\left[\sigma_k\left(\mu(D^2v_t)\right)\right]^{1/k}\ge\left(C_n^k\right)^{1/k}\left(\sigma_n\left(\mu(D^2v_t)\right)\right)^{1/n}\\&=\left(C_n^k\right)^{1/k}\gamma_*=\sup_{\mathbb{R}^n}f\ \ge\ f,|x|>0.\end{aligned}$$

Fix $t_0>-1$ such that $\gamma_0:=\rho(t_0)\ge\gamma_1.$ For any $\gamma>\gamma_0$ and $x\in\Omega,$ let $S_{\gamma,x}$ be the set of (A,k) -convex functions $v\in C(\Omega)$ which is the viscosity subsolution to the problem

$$\begin{cases}\left[\sigma_k\left(\mu(D^2v(y)-A(y,v(y),Dv(y)))\right)\right]^{1/k}=f(y), & y\in\Omega,\\ v(y)=\psi(y), & y\in\partial\Omega,\end{cases}$$

and for any $y\in\Omega,|y-x|\le 2\mathrm{diam}\Omega,$

$$v(y)\le\frac{\gamma_*}{2}|y|^2+\gamma.$$

Then, for all $\rho^{-1}(\gamma_0)<t<\rho^{-1}(\gamma),$ it is clear that the function v_t shown above satisfies $v_t\in S_{\gamma,x},$ or $S_{\gamma,x}\ne\varnothing.$

We define the function

$$u_\gamma(x)=\sup\{w(x):w\in S_{\gamma,x}\},\ x\in\Omega.$$

We prove that u_γ can be extended continuously to $\overline{\Omega}$ and $u_\gamma=\psi$ on $\partial\Omega.$ Indeed, by the Lemma 1 in [13], after extending $\psi\in C^2(\partial\Omega)$ to $\psi\in C^2(\overline{\Omega}),$ there exists a constant $\gamma=\gamma(n,\psi,\Omega)$ such that for any $\eta\in\partial\Omega,$ there exists $\bar{x}(\eta)\in\mathbb{R}^n,|\bar{x}(\eta)|\le\gamma,$ for which function

$$w_\eta(x):=\psi(\eta)+\frac{1}{2}\left(|x-\bar{x}(\eta)|^2-|\eta-\bar{x}(\eta)|^2\right)$$

satisfying $w_\eta<\psi$ in $\overline{\Omega}\setminus\{\eta\}.$ Therefore, we can fix some constant γ_1 such that for any $\eta\in\partial\Omega,$

$$w_\eta(x) \le \frac{\gamma_*}{2}|x|^2 + \gamma_1, \quad \text{dist}(x, \partial\mathfrak{O}) \le 1, \quad x \in \overline{\mathfrak{O}}. \tag{7}$$

By (7), for $\bar{\eta} \in \partial\mathfrak{O}$ and x sufficiently close to $\bar{\eta}$, $x \in \mathfrak{O}$, we have $w_{\bar{\eta}} \in S_{\gamma,x}$. Therefore, $u_\gamma(x) \geqslant w_{\bar{\eta}}(x)$ for x sufficiently close to $\bar{\eta}$. Thus,

$$\liminf_{x\to\bar{\eta}} u_\gamma(x) \geqslant \liminf_{x\to\bar{\eta}} w_{\bar{\eta}}(x) = \psi(\bar{\eta}).$$

From the definition of u_γ we have

$$\limsup_{x\to\bar{\eta}} u_\gamma(x) \leqslant \psi(\bar{\eta}),$$

therefore $\lim_{x\to\bar{\eta}} u_\gamma(x) = \psi(\bar{\eta})$.

We now prove u_γ satisfies (1) in the viscosity sense. By the definition, u_γ is a viscosity subsolution to (1). We only need to prove that u_γ is a viscosity supersolution to (1).

For any $x \in \mathfrak{O}$, fix $0 < \delta < 2\ \text{diam}\Omega$ such that

$$B = B_\delta(x) \subset \mathfrak{O}.$$

From Theorem 2.7, the Dirichlet problem

$$\begin{cases} \left[\sigma_k\left(\mu\left(D^2\tilde{u}(y) - A(y, \tilde{u}(y), D\tilde{u}(y))\right)\right)\right]^{1/k} = f(y), & y \in B, \\ \tilde{u}(y) = u_\gamma(y), & y \in \partial B \end{cases} \tag{8}$$

has a unique $(A,k)-$convexity viscosity solution $\tilde{u} \in C^0(\bar{B})$. By the comparison principle, $u_\gamma \leqslant \tilde{u}$ in B Define

$$\tilde{w}(y) = \begin{cases} \tilde{u}(y), & y \in B \\ u_\gamma(y), & y \in \left(\mathbb{R}^n \setminus \Omega\right) \setminus B, \end{cases}$$

then $\tilde{w} \in S_{\gamma,x}$. Indeed, by the definition of u_γ,

$$u_\gamma(y) \leqslant \frac{\gamma_*}{2}|y|^2 + \gamma, \quad y \in B.$$

Let

$$\tilde{v}(y) = \frac{\gamma_*}{2}|y|^2 + \gamma.$$

Then, for all $y \in B$,

$$\left[\sigma_k\left(\lambda\left(\mathrm{D}^2\tilde{v}(y) - A(y, \tilde{v}(y), D\tilde{v}(y))\right)\right)\right]^{1/k} \ge \left[\sigma_k\left(\lambda\left(\mathrm{D}^2\tilde{v}(y)\right)\right)\right]^{1/k} = \sup_{\mathbb{R}^n} f(y) \ge f(y),$$

$$\tilde{u}(y) = u_\gamma(y) \leqslant \tilde{v}(y), \quad y \in \partial B.$$

From the comparison principle, for any $y \in B$, $\tilde{u} \leqslant \tilde{v}$ i.e., $\tilde{u}(y) \leqslant \frac{\gamma_*}{2}|y|^2 + \gamma.$

By Lemma 2.5,

$$\left[\sigma_k\left(\mu\left(\mathrm{D}^2\tilde{w}(y) - A(y, \tilde{w}(y), D\tilde{w}(y))\right)\right)\right]^{1/k} \geqslant f(y), \quad y \in \mathfrak{D}.$$

Therefore, $\tilde{w} \in S_{\gamma,x}$. And thus, by the definition of $u_\gamma, u_\gamma \geqslant \tilde{w}$ in $\mathfrak{D}$ and $u_\gamma \geqslant \tilde{u}$ in B. Hence,

$$u_\gamma \equiv \tilde{u}, \quad y \in B. \tag{9}$$

However, $\tilde{u}$ satisfies (8), we have, in the viscosity sense,

$$\left[\sigma_k\left(\mu\left(D^2 u_\gamma(y) - A(y, u_\gamma(y), Du_\gamma(y))\right)\right)\right]^{1/k} = f(y), \quad y \in B.$$

Because x is arbitrary, we know that u_γ is a viscosity supersolution of (1).

We prove that u_γ satisfies (5). By the definition of u_γ, $u_\gamma(x) \leqslant \frac{\gamma_*}{2}|x|^2 + \gamma, \quad x \in \mathfrak{D}.$ Then

$$u_\gamma(x) - \frac{\gamma_*}{2}|x|^2 - \gamma \leqslant 0 \leqslant \frac{1}{|x|^{n-2}}, \quad x \in \mathfrak{D}. \tag{10}$$

Moreover, from (6), we have $w_t(x) = \frac{\gamma_*}{2}|x|^2 + \rho(t) - O\left(|x|^{2-n}\right)$ as $|x| \to \infty$. Since $w_t \in S_{\gamma,x}$,

$$u_\gamma(x) - \frac{\gamma_*}{2}|x|^2 - \rho(t) \geqslant -O\left(|x|^{2-n}\right), \text{ as } |x| \to \infty.$$

Let $t \to \mu^{-1}(\gamma)$, we obtain

$$u_\gamma(x) - \frac{\gamma_*}{2}|x|^2 - \gamma \geqslant -O\left(|x|^{2-n}\right). \tag{11}$$

Hence, from (15) and (16), we have

$$\left|u_\gamma(x) - \left(\frac{\gamma_*}{2}|x|^2 + \gamma\right)\right| \leqslant \frac{C}{|x|^{n-2}},$$

for some constant C. Thus,

$$\limsup_{x\to\infty}\left(|x|^{n-2}\left|u_\gamma(x) - \left(\frac{\gamma_*}{2}|x|^2 + \gamma\right)\right|\right) < \infty.$$

Next, we show the uniqueness. Assume that u and v satisfy (1), (2) and (5). From the comparison principle of viscosity solutions to Hessian equations and

$$\lim_{x\to\infty}(u(x) - v(x)) = 0$$

we know $u \equiv v$ in $\mathfrak{D}$. This completes the proof.

Example 2.9. Let $\Omega = B = B_1(0)$ be the unit ball in $\mathbb{R}^n$, $A = -\alpha I\ (\alpha \geq 1)$, $f(x) = (|x|+1)^{-1/2}$, $\psi \in C^2(\partial B)$. Then, all the assumptions of Theorem 2.8 are satisfied. Therefore, there exists γ_0 such that for any $\gamma > \gamma_0$, the problem

$$\begin{cases} [\sigma_k(\mu(D^2u(x) - \alpha \mathrm{I}))]^{1/k} = (|x|+1)^{-1/2}, & x \in \mathbb{R}^n \setminus \overline{B}, \\ u(x) = \psi(x), & x \in \partial B \end{cases}$$

has a unique viscosity solution such that

$$\limsup_{x\to\infty} \left(|x|^{n-2} \left| u(x) - \left(\frac{\gamma_*}{2}|x|^2 + \gamma \right) \right| \right) < \infty,$$

where $\gamma_* = \left(\dfrac{1}{C_n^k} \right)^{1/k}$.

Indeed, it is sufficient to verify the assumptions (AF_3), (AF_4) and (AF_5).

First, we have $\det(-A(x,z,p)) = \alpha^n \geq 1 \geq [f(x)]^k$, or ($AF_3$) is satisfied. Next,

$$\begin{aligned} f(x) - f(y) &= \frac{1}{\sqrt{|x|+1}} - \frac{1}{\sqrt{|y|+1}} \\ &= \frac{|y|-|x|}{\sqrt{|x|+1}\sqrt{|y|+1}(\sqrt{|x|+1}+\sqrt{|y|+1})} \leq \frac{1}{2}|x-y|, \end{aligned}$$

so (AF_4) is satisfied. Moreover, the Dirichlet problem

$$\begin{cases} \Delta v(x) = \mathrm{Tr}(A(x, v(x), Dv(x)) = -n\alpha, & x \in B, \\ v(x) = b, \ x \in \partial B \end{cases}$$

has a classical solution, or (AF_5) holds.

Conclusions

In this paper, we have proved the uniqueness of the solution viscosity solutions in exterior domains of the k-Hessian equations. Our results are significantly extended compared with the findings of the previous studies [8], [12]–[14], [20]. Specifically, we have broadened the class of equations by adding the function A and considering the right-hand side with the bounded function f instead of the constant function 1.

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