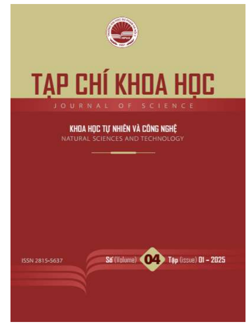
HPU2. Nat. Sci. Tech. Vol 04, issue 01 (2025), 20-30.

HPU2 Journal of Sciences:
Natural Sciences and Technology

Journal homepage: https://sj.hpu2.edu.vn



Article type: Research article

A Lagrange function approach to study second-order optimality conditions for infinite-dimensional optimization problems

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Abstract

In this paper, we focus on the second-order optimality conditions for infinite-dimensional optimization problems constrained by generalized polyhedral convex sets. Our aim is to further explore the role of the generalized polyhedral convex property, which is inspired by the findings of other authors. To this end, we employ the concept of Fréchet second-order subdifferential, a tool in variational analysis, to establish optimality conditions. Furthermore, by applying this concept to the Lagrangian function associated with the problem, we are able to derive refined optimality conditions that surpass existing results. The unique properties of generalized polyhedral convex sets play a crucial role in enabling these improvements.

Keywords: Constrained optimization problem, Second-order necessary condition, Second-order sufficient condition, Fréchet second-order subdifferential, Generalized polyhedral convex set

1. Introduction

First- and second-order optimality conditions are fundamental and intriguing topics in both finite-dimensional and infinite-dimensional mathematical programming. Due to their critical role in both theoretical developments and practical applications, these conditions have attracted significant research interest [1]–[8]. Many researchers have sought to extend these conditions to more general settings, as seen in [9]–[13] and the references therein. It is recognized that first-order and second-order optimality conditions are essential tools for solving optimization problems. In addition, the theory of optimality conditions, especially second-order conditions, is closely linked with sensitivity analysis, see, e.g., [3]

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https://doi.org/10.56764/hpu2.jos.2024.4.1.20-30

Received date: 22-11-2024 ; Revised date: 25-12-2024 ; Accepted date: 25-3-2025

and [8]. In other words, various results concerning optimality conditions were obtained as products of research on sensitivity analysis. Moreover, second-order analysis is crucial in studying the convergence properties of algorithms for solving optimization problems.

Bonnans and Shapiro [3] first introduced the concept of generalized polyhedral convex sets. In a topological vector space X, a set Ω is considered a generalized polyhedral convex set if it can be expressed as the intersection of a finite collection of closed half-spaces and a closed affine subspace. When the affine subspace encompasses the entire space, the set is specifically termed a polyhedral convex set. While these two concepts are identical in finite-dimensional spaces, they exhibit distinct characteristics in infinite-dimensional spaces, where generalized polyhedral convex sets do not reduce to the standard polyhedral convex sets. Several applications of generalized polyhedral convex sets in Banach space settings can be found in the works by Ban et al. [14] and [15]. The theories of generalized linear programming along with quadratic programming in [3], [16]–[17] are mainly based on this concept. The optimization problems discussed in this paper involving generalized polyhedral convex constraint sets are important in optimization theory (see, for example, [18], where full stability of the local minimizers of such problems was characterized). For further information on the properties of generalized polyhedral convex sets and generalized polyhedral convex functions (the functions whose epigraphs are generalized polyhedral convex sets), we refer the interested reader to [19] and [20].

The main goal of this paper is to investigate second-order optimality conditions for infinite-dimensional optimization problems, where the constraint set is generalized polyhedral convex. Our aim to explore more about the role of the generalized polyhedral convex property is inspired by the findings presented in the paper [1]. In addition, Lagrange multipliers have been widely used to establish optimality conditions in problems with constraints, making it interesting to explore how their application and significance have been understood from various perspectives.

The paper is organized as follows: Section 2 provides the foundational groundwork by introducing essential definitions and auxiliary results. Section 3 delves into the core results of the paper, and the concluding section summarizes the key findings.

2. Preliminaries

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be Banach spaces over the field of real numbers. The corresponding duals of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are denoted by $\boldsymbol{X}^*$ and $\boldsymbol{Y}^*$. Let $\boldsymbol{A}$ be a nonempty subset of $\boldsymbol{X}$. The set $\boldsymbol{A}$ is said to be a cone if $\boldsymbol{\alpha A} \subset \boldsymbol{A}$ for any $\boldsymbol{\alpha} > \boldsymbol{0}$. Here, we abbreviate conv $\boldsymbol{A}$ for the convex hull of $\boldsymbol{A}$. In the notation of [21], we write the smallest convex cone containing $\boldsymbol{A}$ to cone $\boldsymbol{A}$. Then, cone $\boldsymbol{A} = \{\boldsymbol{tx} \mid \boldsymbol{t} > \boldsymbol{0}, \boldsymbol{x} \in \text{conv } \boldsymbol{A}\}$. Let $\mathbb{N}$ denote the set of positive integers. Given a linear operator $\boldsymbol{T}$ between Banach spaces, the notation ker $\boldsymbol{T}$ and rge $\boldsymbol{T}$ represent the kernel and the range of $\boldsymbol{T}$, respectively.

Firstly, we recall the concept of contingent cone which our second-order optimality conditions are based on.

Definition 2.1. (See [8]) Let $\boldsymbol{A} \subset \boldsymbol{X}$ and $\boldsymbol{x} \in \boldsymbol{A}$. A direction $\boldsymbol{v}$ is called tangent to $\boldsymbol{A}$ at $\boldsymbol{x}$ if there exist sequences of points $\boldsymbol{x_k} \in \boldsymbol{A}$ and scalars $\boldsymbol{t_k}$, $\boldsymbol{t_k} \geq \boldsymbol{0}$, $\boldsymbol{k} \in \mathbb{N}$, such that $\boldsymbol{t_k} \to \boldsymbol{0}^+$and $\boldsymbol{v} = \lim_{\boldsymbol{k}\to\infty} \left[\boldsymbol{t_k^{-1}}(\boldsymbol{x_k} - \boldsymbol{x})\right]$.

The set of all tangent directions to A at x, denoted by $T(A, x)$, is called the contingent cone (or the Bouligand-Severi tangent cone, see [22]) to A at x.

Remark 2.1. From the definition, it is not hard to show that a vector $v \in T(A,x)$ if and only if we can find a sequence $\{t_k\}$ of positive scalars and a sequence of vectors $\{v_k\}$ with $t_k \to 0^+$and $v_k \to v$ as $k \to \infty$ such that $x_k := x + t_k v_k$ belongs to A for all $k \in \mathbb{N}$.

Secondly, we show the concept of the generalized polyhedral convex set which is the main objective to study in this paper.

Definition 2.2. (See [3] and [21]) A subset $A \subset X$ is called a generalized polyhedral convex set if there exist $u_i^* \in X^*$, real numbers $\beta_i, i = 1,2,\dots,p$, and a closed affine subspace $L \subseteq X$, such that

$$A = \{u \in X \mid u \in L, \langle u_i^*, u\rangle \le \beta_i, i = 1,2,\dots,p\}. \tag{1}$$

In the case $L = X$, one says that A is a polyhedral convex set.

Remark 2.2. (i) Every generalized polyhedral convex set is closed.

(ii) If X is finite-dimensional, it has been shown in [21] that a subset $A \subset X$ is generalized polyhedral convex if and only if it is polyhedral convex.

Let the generalized polyhedral convex set A be given in (1). By [3], there exist a continuous surjective linear mapping T from X to a Banach space Y and a vector $v \in Y$ such that $L = \{u \in X \mid Tu = v\}$. So,

$$A = \{u \in X \mid Tu = v, \langle u_i^*, u\rangle \le \beta_i, i = 1,2,\dots,p\}. \tag{2}$$

Set $I = \{1,2,..,p\}$ and $I(u) := \{i \in I \mid \langle u_i^*, u\rangle = \beta_i\}$ for any $u \in A$. From now on, our work will focus on the generalized polyhedral convex set A which has the form as in (2).

Given a point $\bar{u} \in A$, the following proposition gives the formula for computing the tangent cone to the generalized polyhedral convex set A at $\bar{u}$.

Proposition 2.1. (See, e.g., [1], [14]) Let X be a Banach space, A be a generalized polyhedral convex set in X. Given $\bar{u} \in A$. The tangent cone to A at $\bar{u}$ is

$$T(A,\bar{u}) = \{v \in X \mid Tv = 0, \langle u_i^*, v\rangle \le 0, i \in I(\bar{u})\}.$$

Lastly, we recall the Fréchet second-order subdifferential concept and some related constructions from the book [22].

Definition 2.3. (See [22]) Let A be a nonempty subset of X. For any $\bar{u} \in A$, the Fréchet normal cone of A at $\bar{u}$ is defined by

$$N(A,\bar{u}) := \left\{u^* \in X^* \;\middle|\; \limsup_{u \xrightarrow{\Omega} \bar{u}} \frac{\langle u^*, u - \bar{u}\rangle}{\|u - \bar{u}\|} \le 0\right\},$$

where $u \xrightarrow{A} \bar{u}$ means that $u \to \bar{u}$ and $u \in A$. If $\bar{u} \notin A$, one says that the set $N(A,\bar{u})$ is an empty set.

If A is a convex set in X, then by [22, Proposition 1.5], one has

$$N(A,\bar{u}) := \{u^* \in X^* \mid \langle u^*, u - \bar{u}\rangle \le 0, \forall u \in A\},$$

that is the Fréchet normal cone of A at $\bar{u}$ coincides with the normal cone in the sense of convex analysis.

Let $G: X \rightrightarrows Y$ be a set-valued map with the graph

$$\mathrm{gph}G = \{\, (u,v) \in X \times Y \mid v \in G(u) \,\}.$$

The product space $X \times Y$ is equipped with the norm $\|(u,v)\| := \|u\| + \|v\|$.

Definition 2.4. (See [22]) Given $(u,v) \in \text{gph}G$. The Fréchet coderivative of G at (u,v) is a multifunction $D^*G(u,v): Y^* \rightrightarrows X^*$ given by

$$D^*G(u,v)(v^*) := \{u^* \in X^* \mid (u^*, -v^*) \in N(\text{gph}G, (u,v))\}, \forall v^* \in Y^*.$$

If $(u,v) \notin \text{gph}G$ then one puts $D^*G(u,v)(v^*) = \emptyset$ for any $v^* \in Y^*$. We omit $v = g(u)$ in the above coderivative notion if $G = g: X \to Y$ is a single valued map, i.e., we will write $Dg(u)(v^*)$ instead of $D^*g(u, g(u))(v^*)$. If $g: X \to Y$ is Fréchet differentiable at u. Then by [22] one has $Dg(u)(v^*) = \{\nabla g(u)^* v^*\}$ for every $v^* \in Y^*$, where $\nabla g(u)^*$ is the adjoint operator of $\nabla g(u)$. This formula shows that the coderivative under consideration is an appropriate extension of the adjoint derivative operator of the real-valued function to the case of the set-valued map.

Consider a function $h: X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$ is the extended-real line. The epigraph of h is given by epi $h = \{(u,t) \in X \times \mathbb{R} \mid t \geq h(u)\}$.

Definition 2.5. (See, e.g., [22]) Let $h: X \to \overline{\mathbb{R}}$ be finite at a point u. The Fréchet subdifferential of h at u is given by

$$\partial h(u) := \{u^* \in X^* \mid (u^*, -1) \in N(\text{epi}h, (u, h(u)))\}.$$

If $|h(u)| = \infty$ then we put $\partial h(u) = \emptyset$.

Throughout the paper, $h \in \mathcal{C}^1$ is understood that it is continuously Fréchet differentiable and its gradient mapping ∇h is continuous. Similarly, $h \in \mathcal{C}^2$ means that h is twice continuously differentiable. If h is a $\mathcal{C}^1$ function, then for any u with $|h(u)| < \infty$, the Fréchet subdifferential contains only the gradient $\{\nabla h(u)\}$ (see [22]).

One can use the notion of coderivative to define the second-order subdifferential of extended-real-valued functions.

Definition 2.6. (See [22]) Let $h: X \to \overline{\mathbb{R}}$ be a function with a finite value at u. For any $v \in \partial h(u)$, the mapping $\partial^2 h(u,v): X^{**} \rightrightarrows X^*$ with the values

$$\partial^2 h(u,v)(v^*) := (D^*\partial h)(u,v)(v^*) = \{u^* \in X^* \mid (u^*, -v^*) \in N(\text{gph}\partial h, (u,v))\}$$

is called the Fréchet second-order subdifferential of h at u relative to v.

The symbol v in the notation $\partial^2 h(u,v)(v^*)$ will be omitted if $\partial h(u)$ is a singleton. Moreover, if h is Fréchet differentiable at u, then $Dh(u)(v) = \{\nabla h(u)^* v\}$ for every $v \in X$ as was mentioned after the definition of coderivative. In addition, if $h \in \mathcal{C}^2$ around u, i.e. h is twice continuously differentiable in an open neighborhood of u, then from the above fact and Definition 2.6, one has

$$\partial^2 h(u)(v^*) = \{\nabla^2 h(u)^* v^*\} \; \forall v^* \in X^{**}$$

where $\nabla^2 h(u)^*$ is the adjoint operator of the second-order derivative $\nabla^2 h(u)$ of h at u. In particular, when X is finite-dimensional, $\nabla h(u)$ reduces to the classical Hessian matrix for which $\nabla^2 h(u)^* = \nabla^2 h(u)$.

3. Main results

In this paper, we consider the optimization problem

$$\min\{f(x) \mid x \in \Omega\} \tag{P}$$

where $f: X \to \mathbb{R}$ is a $\mathcal{C}^1$ function and $\Omega \subset X$ is a generalized polyhedral convex set.

The Lagrange function corresponding to problem (P) is

$$\mathcal{L}(x, v^*, \lambda) = f(x) + \langle v^*, Tx - y\rangle + \sum_{i=1}^{p} \lambda_i(\langle u_i^*, x\rangle - \beta_i)$$

where $\lambda_1, \lambda_2, \dots, \lambda_p$ are real numbers and $v^* \in Y^*$.

The first-order necessary optimality conditions have been previously studied. As the statement in the general setting is given in [23] without proof, we will present the proof for our case in detail for the reader's convenience.

Theorem 3.1. *Let Ω be a generalized polyhedral convex set given by (2) and $\bar{x}$ be a local solution of (P). Then there exist multipliers $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_p) \in \mathbb{R}^p, \bar{\lambda}_i \geq 0$ and $\bar{v}^* \in Y^*$ such that*

$$\begin{cases} \mathcal{L}_x(\bar{x}, \bar{v}^*, \bar{\lambda}) = \nabla f(\bar{x}) + T^*\bar{v}^* + \sum_{i=1}^{p} \bar{\lambda}_i u_i^* = 0, \\ \bar{\lambda}_i[\langle u_i^*, \bar{x}\rangle - \beta_i] = 0, i \in I, \end{cases} \tag{3}$$

where $\mathcal{L}_x$ denotes the partial derivative of $\mathcal{L}$ with respect to the variable x.

Proof. Let $\bar{x}$ be a local solution of (P) and Ω be given by (2). Noting that Ω is convex, by Proposition 5.1 in [24], we have

$$-\nabla f(\bar{x}) \in N(\Omega, \bar{x}). \tag{4}$$

Since Ω is generalized polyhedral, it follows that its normal cone is generalized polyhedral as well. Thanks to [21], one gets

$$N(\Omega, \bar{x}) = \text{cone}\{u_i^* \mid i \in I(\bar{x})\} + (\ker T)^\perp \tag{5}$$

with $(\ker T)^\perp$ being the annihilator of the linear subspace $\ker T$, i.e.,

$$(\ker T)^\perp = \{x^* \in X^* \mid \langle x^*, v\rangle = 0, \forall v \in \ker T\}.$$

Combining (4) with (5) implies that there exist multipliers $(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_p) \in \mathbb{R}^p, \bar{\lambda}_i \geq 0$, $i \in I(\bar{x})$ satisfying

$$\nabla f(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i u_i^* \in -(\ker T)^\perp. \tag{6}$$

Moreover, since T is surjective, by invoking the lemma on the annihilator [23] (see also [25]), one has $(\ker T)^\perp = \text{rge} T^*$. Hence, $x^* \in -(\ker T)^\perp$ if and only if we can find $\bar{v}^* \in Y^*$ satisfying $x^* = -T^*\bar{v}^*$. Thus (6) yields the existence of a vector $\bar{v}^* \in Y^*$ such that

$$\nabla f(\bar{x}) + T^*\bar{v}^* + \sum_{i=1}^{p} \bar{\lambda}_i u_i^* = 0, \forall i \in I(\bar{x}).$$

Consequently, by choosing $\bar{\lambda}_i = 0$ for all $i \in I \setminus I(\bar{x})$, we obtain a multiplier set $\bar{\lambda}_i \geq 0$, $i = 1,2, \dots, p$ and $\bar{v}^* \in Y^*$ such that (3) holds for every $i \in I$. □

Remark 3.1. The multipliers $\bar{\lambda}$ and functional $\bar{v}^*$ in Theorem 3.1 are referred to as the Lagrange multipliers. If $\bar{x}$ is a feasible point of (P) and there exists $(v^*, \lambda) \in Y^* \times \mathbb{R}^p$ satisfying (3), then $\bar{x}$ is called a stationary point of (P). The set of Lagrange multipliers of (P) at $\bar{x} \in \Omega$ is denoted by $\Lambda(\bar{x})$.

The following theorem provides the second-order necessary optimality condition for the problem (P). This optimality condition is formulated using the second-order Fréchet subdifferential of the Lagrange function.

Theorem 3.2. (Second-order necessary optimality conditions) *Let X be a Banach space. Suppose that $\bar{x}$ is a local solution of (P), $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_p) \in \mathbb{R}^p, \bar{\lambda} \geq 0$ and $\bar{v}^*$ is the Lagrange multiplier corresponding to $\bar{x}$, that is (3) holds. Assume further that there is a constant $l > 0$ satisfying*

$$\|\nabla f(x) - \nabla f(\bar{x})\| \leq l\|x - \bar{x}\| \tag{7}$$

for every x in a neighborhood U of $\bar{x}$. Then, for any $u \in T(\Omega, \bar{x})$ such that $-u \in T(\Omega, \bar{x})$, and $\langle \nabla f(\bar{x}), u\rangle = 0$, the inclusion

$$z \in \partial^2 \mathcal{L}(\cdot, \bar{v}^*, \bar{\lambda}) \left(\bar{x}, \mathcal{L}_x(\bar{x}, \bar{v}^*, \bar{\lambda})\right)(u) \tag{8}$$

implies that

$$\langle z, u\rangle \geq 0. \tag{9}$$

Proof. Suppose that $\bar{x}$ is a local solution of (P) and (7) holds for all x in a neighborhood U of $\bar{x}$, with $l > 0$. Let $u \in T(\Omega, \bar{x})$ be such that $-u \in T(\Omega, \bar{x})$ and $\langle \nabla f(\bar{x}), u\rangle = 0$. To obtain a contradiction, assume that there exists $z \in X^*$ satisfying (8) such that $\langle z, u\rangle < 0$. We first observe from (3) that $\mathcal{L}_x(\bar{x}, \bar{v}^*, \bar{\lambda}) = 0$. Consequently, (8) means that $z \in \partial^2 \mathcal{L}(\cdot, \bar{v}^*, \bar{\lambda})((\bar{x}, 0))(u)$. By the definition of the Fréchet second-order subdifferential, the latter is equivalent to $z \in D^* \mathcal{L}_x(\cdot, \bar{v}^*, \bar{\lambda})((\bar{x}, 0))(u)$, or, equivalently, $(z, -u) \in N(\text{gph}\mathcal{L}_x(\cdot, \bar{v}^*, \bar{\lambda}), (\bar{x}, 0))$. Therefore, one has

$$\limsup_{x \to \bar{x}} \frac{\left\langle (z, -u), \left(x, \mathcal{L}_x(x, \bar{v}^*, \bar{\lambda})\right) - (\bar{x}, 0)\right\rangle}{\|x - \bar{x}\| + \left\|\mathcal{L}_x(x, \bar{v}^*, \bar{\lambda})\right\|} \leq 0. \tag{10}$$

We recall that every vector $v \in X$ can be regarded as an element of x^{**} by setting $\langle v, x^*\rangle = \langle x^*, v\rangle$ for all $x^* \in X^*$. Hence $\langle u, \mathcal{L}_x(x, \bar{v}^*, \bar{\lambda})\rangle = \langle \mathcal{L}_x(x, \bar{v}^*, \bar{\lambda}), u\rangle$ for every $u, x \in X$. So, by rearranging (10) one obtains

$$\limsup_{x \to \bar{x}} \frac{\langle z, x - \bar{x}\rangle - \langle \mathcal{L}_x(x, \bar{v}^*, \bar{\lambda}), u\rangle}{\|x - \bar{x}\| + \left\|\mathcal{L}_x(x, \bar{v}^*, \bar{\lambda})\right\|} \leq 0. \tag{11}$$

By our assumptions, both vector u and $-u$ belong to the tangent cone $T(\Omega, \bar{x})$. Moreover, since Ω is generalized polyhedral convex, one can find $\bar{k} \in \mathbb{N}$ such that $x_k := \bar{x} - \frac{1}{k}u$ belongs to Ω for every $k \geq \bar{k}$. By applying the classical mean value theorem (see [26]) for the real continuous function

$$\Phi(\mu) := \mathcal{L}((1 - \mu)\bar{x} + \mu x_k, \bar{v}^*, \bar{\lambda})$$

where $\mu \in [0,1]$ on the interval $[0,1]$ and using the chain rule (see [27]), we find $\xi_k \in (\bar{x}, x_k) := \{(1 - \mu)\bar{x} + \mu x_k \mid \mu \in (0,1)\}$ such that

$$\mathcal{L}(x_k, \bar{v}^*, \bar{\lambda}) - \mathcal{L}(\bar{x}, \bar{v}^*, \bar{\lambda}) = \langle \mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda}), x_k - \bar{x}\rangle. \tag{12}$$

On one hand, by the first equality in (3), for each $u \in T(\Omega, \bar{x})$ we have

$$0 = \langle \mathcal{L}_x(\bar{x}, \bar{v}^*, \bar{\lambda}), u\rangle$$

$$= \left\langle \nabla f(\bar{x}) + T^*\bar{v}^* + \sum_{i=1}^{p} \bar{\lambda}_i u_i^*, u \right\rangle$$

$$= \langle \nabla f(\bar{x}), u\rangle + \langle T^*\bar{v}^*, u\rangle + \left\langle \sum_{i=1}^{p} \bar{\lambda}_i u_i^*, u \right\rangle$$

$$= \langle \nabla f(\bar{x}), u\rangle + \langle \bar{v}^*, Tu\rangle + \left\langle \sum_{i=1}^{p} \bar{\lambda}_i u_i^*, u \right\rangle.$$

As $u \in T(\Omega, \bar{x})$ and $\langle \nabla f(\bar{x}), u\rangle = 0$, the latter implies

$$\left\langle \sum_{i=1}^{n} \bar{\lambda}_i u_i^*, u \right\rangle = 0. \tag{13}$$

On the other hand, thanks to (3) and the fact that both x_k and $\bar{x}$ belong to Ω, one has

$$\mathcal{L}(x_k, \bar{v}^*, \bar{\lambda}) - \mathcal{L}(\bar{x}, \bar{v}^*, \bar{\lambda}) = f(x_k) + \langle \bar{v}^*, Tx_k - y\rangle + \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x_k\rangle - \beta_i)$$

$$- f(\bar{x}) - \langle \bar{v}^*, T\bar{x} - y\rangle - \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, \bar{x}\rangle - \beta_i)$$

$$= f(x_k) - f(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x_k\rangle - \beta_i).$$

Keeping in mind that $x_k = \bar{x} - \frac{1}{k}u$, by (13) we can assert that

$$\sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x_k\rangle - \beta_i) = \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, \bar{x}\rangle - \beta_i) - \frac{1}{k}\sum_{i=1}^{p} \bar{\lambda}_i\langle u_i^*, u\rangle = 0.$$

Consequently, $\mathcal{L}(x_k, \bar{v}^*, \bar{\lambda}) - \mathcal{L}(\bar{x}, \bar{v}^*, \bar{\lambda}) = f(x_k) - f(\bar{x}) \geq 0$, because $\bar{x}$ is a local minimum point of (P). Combining the latter with (12) and noting that $x_k = \bar{x} - \frac{1}{k}u$ we arrive at

$$\langle \mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda}), u\rangle \leq 0. \tag{14}$$

By (11) and the fact that $\xi_k \to \bar{x}$ as $k \to \infty$, one gets

$$\limsup_{x \to \bar{x}} \frac{\langle z, \xi_k - \bar{x}\rangle - \langle \mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda}), u\rangle}{\|\xi_k - \bar{x}\| + \|\mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda})\|} \leq 0. \tag{15}$$

Put

$$\Delta_k := \frac{\langle z, \xi_k - \bar{x}\rangle - \langle \mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda}), u\rangle}{\|\xi_k - \bar{x}\| + \|\mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda})\|}.$$

By (14), we get the following estimate for Δ_k

$$\Delta_k \geq \frac{\langle z, \xi_k - \bar{x}\rangle}{\|\xi_k - \bar{x}\| + \|\mathcal{L}_x(\xi_k, \bar{v}^*, \bar{\lambda})\|}.$$

Observing that $\xi_k = \bar{x} - t_k u$, for some $t_k \in \left(0, \frac{1}{k}\right)$, the latter implies that

$$\Delta_k \geq \frac{-t_k\langle z, u\rangle}{\|t_k u\| + \left\|\mathcal{L}_x\left(\xi_k, \bar{v}^*, \bar{\lambda}\right)\right\|}$$
$$= \frac{-\langle z, u\rangle}{\|u\| + t_k^{-1}\left\|\mathcal{L}_x\left(\xi_k, \bar{v}^*, \bar{\lambda}\right)\right\|}.$$

On one hand, by virtue of (3), we have

$$\left\|\mathcal{L}_x\left(\xi_k, \bar{v}^*, \bar{\lambda}\right)\right\| = \left\|\nabla f(\xi_k) + T^*\bar{v}^* + \sum_{i=1}^{p} \bar{\lambda}_i u_i^* - \nabla f(\bar{x}) - T^*\bar{v}^* - \sum_{i=1}^{p} \bar{\lambda}_i u_i^*\right\|$$
$$= \|\nabla f(\xi_k) - \nabla f(\bar{x})\|.$$

Hence, by (7), we get

$$\left\|\mathcal{L}_x\left(\xi_k, \bar{v}^*, \bar{\lambda}\right)\right\| \leq l\|\xi_k - \bar{x}\| = l t_k\|u\|.$$

Therefore,

$$\Delta_k \geq \frac{-\langle z, u\rangle}{\|u\| + l\|u\|}.$$

Consequently, $\limsup_{k\to\infty} \Delta_k > 0$. This contradicts to (15). The proof is complete. □

We are now in a position to establish second-order sufficient optimality conditions for problem (P).

Theorem 3.3. (Second-order sufficient optimality conditions) *Let X be a reflexive Banach space. Assume that $\bar{x}$ is a stationary point of (P), $\left(\bar{v}^*, \bar{\lambda}\right)$ is the unique Lagrange multiplier from $\Lambda(\bar{x})$. If there exists a constant $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$ one has*

$$\langle z, u\rangle \geq 0 \text{ for all } u \in X \text{ and } z \in \partial^2\mathcal{L}\left(x, \bar{v}^*, \bar{\lambda}\right)(u) \tag{16}$$

then $\bar{x}$ is a local solution of (P).

Proof. Since X is a reflexive Banach space, it follows that X is an Asplund space. Moreover, one has $X^{**} = X$. As

$$x \mapsto \mathcal{L}\left(x, v^*, \bar{\lambda}\right) = f(x) + \langle v^*, Tx - y\rangle + \textstyle\sum_{i=1}^{p} \lambda_i(\langle u_i^*, x\rangle - \beta_i),$$

is a $\mathcal{C}^1$ function on X, it yields from (16) and [28], whose proof can be applied to any open convex subset of the space in question, that $\mathcal{L}\left(., \bar{v}^*, \bar{\lambda}\right)$ is convex on $B(\bar{x}, \delta)$. As $\bar{v}^*, \bar{\lambda} \in \Lambda(\bar{x})$, one has $\mathcal{L}_x\left(\bar{x}, \bar{v}^*, \bar{\lambda}\right) = 0$. Combining this with the convexity of $\mathcal{L}\left(., \bar{v}^*, \bar{\lambda}\right)$ on $B(\bar{x}, \delta)$ yields

$$\mathcal{L}\left(x, \bar{v}^*, \bar{\lambda}\right) - \mathcal{L}\left(\bar{x}, \bar{v}^*, \bar{\lambda}\right) \geq \left\langle \mathcal{L}_x\left(\bar{x}, \bar{v}^*, \bar{\lambda}\right), x - \bar{x}\right\rangle = 0,\ \forall x \in B(\bar{x}, \delta).$$

So

$$\begin{aligned} &f(x) + \langle v^*, Tx - y\rangle + \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x\rangle - \beta_i) - f(\bar{x}) \\ &-\langle v^*, T\bar{x} - y\rangle - \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, \bar{x}\rangle - \beta_i) \geq 0, \forall x \in B(\bar{x}, \delta). \end{aligned} \tag{17}$$

On one hand, $\bar{x} \in C$, one gets $\langle v^*, T\bar{x} - y\rangle = 0$ and $-\sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, \bar{x}\rangle - \beta_i) = 0$ because of (3). On the other hand, one has

$$\mathcal{L}\big(x, \bar{v}^*, \bar{\lambda}\,\big) = f(x) + \sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x\rangle - \beta_i),\ \forall x \in \Omega.$$

Consequently, from (17) we arrive at

$$f(x) + \textstyle\sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x\rangle - \beta_i) - f(\bar{x}) \geq 0, \forall x \in \Omega \cap B(\bar{x}, \delta),$$

or, equivalently,

$$f(x) - f(\bar{x}) \geq -\sum_{i=1}^{p} \bar{\lambda}_i(\langle u_i^*, x\rangle - \beta_i) \geq 0, \forall x \in \Omega \cap B(\bar{x}, \delta). \tag{18}$$

Therefore, (18) implies that $f(x) \geq f(\bar{x})$ for every $x \in \Omega \cap B(\bar{x}, \delta)$. In conclusion, $\bar{x}$ is a local solution of (P). □

Let us consider examples to illustrate our results.

Example 3.1. Let $X = \mathbb{R}^2$. Consider problem (P) with $f(x) = 6x_1 + x_1^2 + 2x_2 + x_2^2$ and $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 120, x_1 \geq 0, x_2 \geq 0\}$. The Lagrangian function is

$$\mathcal{L}(x, v^*, \lambda) = 6x_1 + x_1^2 + 2x_2 + x_2^2 + v^*(x_1 + x_2 - 120) - \lambda_1 x_1 - \lambda_2 x_2,$$

for $x = (x_1, x_2) \in \mathbb{R}^2$ and $v^* \in \mathbb{R}, \lambda_1, \lambda_2 \in \mathbb{R}^+$.

It is not hard to see that this problem has the stationary point $\bar{x} = (59{,}61)$ with the Lagrange multiplier $\lambda_1 = \lambda_2 = 0, v^* = -124$.

Since $\mathcal{L}(.\,, v^*, \lambda)$ is a C^2-function, it follows that $\partial^2\mathcal{L}(x, v^*, \lambda) = \left\{\left(\nabla_x^2\mathcal{L}(x, v^*, \lambda)\right)^*\right\} = \{\nabla_x^2\mathcal{L}(x, v^*, \lambda)\}$ for all $x \in X$ and $v^* \in \mathbb{R}, \lambda_1, \lambda_2, \in \mathbb{R}^+$. For every $u = (u_1, u_2) \in \mathbb{R}^2$, one has

$$\partial^2\mathcal{L}(x, v^*, \lambda)(u) = \left\{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix}\right\}.$$

We now check all the assumptions of Theorem 3.3 at $\bar{x} = (59{,}61), \bar{v}^* = -124$ and $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$. Firstly, for all $x \in X, u \in X$ and $z \in \partial^2\mathcal{L}\big(x, \bar{v}^*, \bar{\lambda}\big)(u)$, we see that

$$\langle z, u\rangle = \left\langle \begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle = 2u_1^2 + 2u_2^2 \geq 0,$$

hence condition (16) is fulfilled. So, $\bar{x}$ is a local solution of (P) by Theorem 3.3. Secondly, by using the second-order necessary optimality conditions in Theorem 3.2, we can verify that the stationary point $\bar{x} = (59{,}61)$ is the local solution of (P). As f is continuous and Ω is a nonempty compact set, it follows that (P) has the global solution by the Weierstrass theorem.

We end this section with an example in an infinite-dimensional space.

Example 3.2. Let ℓ^2 denote the Hilbert space of all square summable real sequences, $\ell^2 = \{x = (x_1, x_2, \ldots, x_n, \ldots) \mid \sum_{n=1}^{+\infty} x_n^2 < +\infty, x_n \in \mathbb{R}, n = 1,2, \ldots.\}$. The scalar product and the norm in ℓ^2 are defined, respectively, by

$$\langle x, y\rangle = \sum_{n=1}^{+\infty} x_n y_n\,, \quad \| x \| = \left(\sum_{n=1}^{+\infty} x_n^2\right)^{\frac{1}{2}}.$$

Consider problem (P) with $f(x) = \frac{1}{2}\langle x, Qx\rangle$ and $\Omega = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2 \mid \langle u_i^*, x\rangle \leq 0, i = 1,2, \ldots, p\}$, where $Q: \ell^2 \to \ell^2$ is defined by $Qx = (x_1, x_2, x_3, \ldots)$ and $u_i^* = (0,0, \ldots, \underbrace{-1}_{i}, 0, \ldots) \in \ell^2,\ i = 1,2, \ldots, p$.

The Lagrangian function is

$$\mathcal{L}(x,\lambda) = \frac{1}{2}\langle x, Qx\rangle + \sum_{i=1}^{p} \lambda_i \langle u_i^*, x\rangle$$

for $x = (x_1, x_2, x_3, \dots) \in \ell^2$ and $\lambda_i \in \mathbb{R}^+, i = 1,2,\dots,p$.

The problem has a stationary point $\bar{x} = (0,0,\dots,0,\dots)$ with Lagrange multiplier $\lambda = 0$. Since $\mathcal{L}(.,\lambda)$ is a C^2-function, we have that $\partial^2\mathcal{L}(x,\lambda) = \{(\nabla_x^2\mathcal{L}(x,\lambda))^*\} = \{\nabla_x^2\mathcal{L}(x,\lambda)\}$ for all $x \in X$ and $\lambda_i \in \mathbb{R}^+, i = 1,2,\dots,p$. For every $u = (u_1, u_2, \dots) \in \ell^2$, one has

$$\partial^2\mathcal{L}(x,\lambda)(u) = Qu.$$

Thus, for every $z \in \partial^2\mathcal{L}(x, \bar{v}^*, \bar{\lambda})(u)$,

$$\langle z, u\rangle = \langle Qu, u\rangle = u_1^2 + u_2^2 + u_3^2 + \cdots \geq 0,$$

hence condition (16) is fulfilled. So, $\bar{x}$ is a local solution of (P) by Theorem 3.3.

4. Conclusion

In this paper, second-order necessary and sufficient optimality conditions for infinite-dimensional optimization problems with generalized polyhedral convex constraints are derived. Second-order necessary optimality conditions are presented in Theorem 3.2 meanwhile second-order sufficient optimality conditions are given in Theorem 3.2. An example to illustrate the obtained results is included as well.

Acknowledgments

The paper was supported by the Science and Technology Fund of the Vietnam Ministry of Education and Training (B2024-TNA-20).

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