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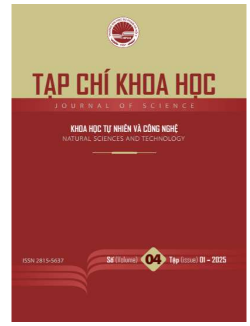

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On the multiplicity of graded fiber cones with arbitrary dimensions

Tien-Manh Nguyen*

Hung Vuong University, Phu Tho, Vietnam

Abstract

Let (A,m) be a Noetherian local ring with maximal ideal m, $J \subset A$ an m-primary ideal, $S = \underset{n\geq 0}{\oplus} S_n$ a finitely generated standard graded algebra over A and $M = \underset{n\geq 0}{\oplus} M_n$ a finitely generated graded S-module. Then $F_J(M) = \bigoplus_{n\geq 0} \dfrac{M_n}{JM_n}$ is called the fiber cone of the graded module M with respect to J. As we know, the concept of multiplicities in commutative algebra is an object that plays an important role in determining the properties and classifying the structure of rings and modules including the Cohen-Macaulay property. In this paper, we establish the multiplicity formula of the fiber cone $F_J(M) = \bigoplus_{n\geq 0} \dfrac{M_n}{JM_n}$ with arbitrary dimensions. The concept we used to prove the results is filter-regular sequences of graded modules. Our approach is based on the formulas of multiplicities of fiber cones of graded modules in the case that the dimensions of those fiber cones equal 1 and by induction on dimensions of fiber cones of graded modules.

Keywords: Noetherian ring, multiplicity, graded module, fiber cone, filter-regular sequence

1. Introduction

Throughout this paper, let (A,m) be a Noetherian local ring with maximal ideal m. Let J be an m-primary ideal of A, $S = \underset{n\geq 0}{\oplus} S_n$ a finitely generated standard graded algebra over A, $S_+ = \underset{n> 0}{\oplus} S_n$ and $M = \underset{n\geq 0}{\oplus} M_n$ a finitely generated graded S-module. Set $F_J(S) = \bigoplus_{n\geq 0} \dfrac{S_n}{JS_n}$, $F_J(M) = \bigoplus_{n\geq 0} \dfrac{M_n}{JM_n}$.

* Corresponding author, E-mail: nguyentienmanh@hvu.edu.vn

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We call $F_J(S)$ *the fiber cone of* S with respect to J and $F_J(M)$ *the fiber cone of* M with respect to J.

In the case that $S = A[It] = \underset{n\geq 0}{\oplus} I^n t^n$, where I is an ideal of A ($A[It]$ is also *the Rees algebra* of I), then $F_J(S) = \bigoplus_{n\geq 0} \dfrac{I^n}{JI^n}$ is called *the fiber cone of* I with respect to J and denote by $F_J(I)$, especially $J = m$ then $F_m(I) = \bigoplus_{n\geq 0} \dfrac{I^n}{mI^n}$ is called *the fiber cone of* I. The multiplicity and the Cohen-Macaulayness of fiber cones have attracted much attention (see [1]–[9], [10]–[15]). In 1988, Huneke and Sally proved that $F_m(I)$ is Cohen-Macaulay if A is a Cohen-Macaulay ring and I is an m-primary with is the reduction number $r(I) \leq 1$ [1]. Next, Shah showed that $F_m(I)$ is Cohen-Macaulay if I is an equimultiple ideal with $grade(I) = ht(I)$ and the reduction number $r(I) \leq 1$ [10], [11]. More generally, when I is an arbitrary ideal of A, by using the concept of weak-(FC)-sequences of ideals in local rings Viet established the multiplicity formula and characterized the Cohen-Macaulayness of $F_m(I)$ [12]. Continuing to expand in this direction, Viet built the concept of weak-(FC)-sequences in graded algebras to establish the multiplicity formula and characterize the Cohen-Macaulayness of fiber cones of graded algebras [13]. In addition, the multiplicity and the Cohen-Macaulayness of fiber cones of good filtrations in local rings were determined by Viet and Thanh in 2011 [14]. In 2021, by using the concept of weak-(FC)-sequences in graded modules, Manh obtained some results on the multiplicity and the Cohen-Macaulayness of $F_J(M)$ [8]. Moreover, Manh gave more ways to express the multiplicity and characterized the Cohen-Macaulayness of $F_J(M)$ in the case that $\dim\left(F_J(M)\right) = 1$ by using the concept of filter-regular sequences [9]. As a continuation, the purpose of this paper is to establish the multiplicity formula and characterize the Cohen-Macaulayness of the fiber cone $F_J(M) = \bigoplus_{n\geq 0} \dfrac{M_n}{JM_n}$ with arbitrary dimensions. The tool used is the concept of filter-regular sequences in graded modules. For convenience of presentation, we set $a : b^\infty := \bigcup_{n\geq 0}^{\infty} (a : b^n)$.

2. Research content

2.1. On filter-regular sequences of graded modules

Regular-filter sequences are an important concept of commutative algebra introduced by the group of authors Nguyen Tu Cuong, Schenzel, and Ngo Viet Trung in 1978 [16] and later mentioned by Stückrad and Vogel in [17].

Definition 2.1 [16]**.** Set $S_+ = \underset{n>0}{\oplus} S_n$. Assume that $S_1 \not\subset \sqrt{\mathrm{Ann}_S(M)}$. Then:

(i) A homogeneous element $x \in S$ is called an S_+*-filter-regular element* with respect to M if $x \notin P$ for any $P \in \mathrm{Ass}_S(M)$, $S_+ \not\subset P$.

(ii) A homogeneous sequence $x_1,\ldots,x_t \in S$ is called an S_+ *-filter-regular sequence* with respect to M if x_i an S_+ *- filter-regular element* with respect to $\dfrac{M}{(x_1,\ldots,x_{i-1})M}$ for all $i=1,\ldots,t.$

Remark 2.2 [15]**.** (i) A homogeneous element $x \in S$ is S_+ *-filter-regular element* with respect to M if and only if satisfies one of two following conditions:

$$(i_1)\ 0_M : x \subseteq 0_M : S_+^{\infty}.$$

(i_2) $(0_M : x)_n = 0$ for all large $n.$

(ii) An S_+ -filter-regular sequence $x_1,\ldots,x_t$ with respect to M is a maximal S_+ -filter-regular sequence if $S_1 \not\subset \sqrt{\mathrm{Ann}_S\left(\dfrac{M}{(x_1,\ldots,x_{t-1})M}\right)}$ and $S_1 \subset \sqrt{\mathrm{Ann}_S\left(\dfrac{M}{(x_1,\ldots,x_t)M}\right)}.$

The following proposition showed that the universal existence of filter-regular sequences.

Proposition 2.3 [15]**.** Assume that $S_1 \not\subset \sqrt{\mathrm{Ann}_S(M)}.$ *Then there exists an* S_+ *-filter-regular element* $x \in S_1$ *with respect to* $M.$

Let a homogeneous ideal $\mathcal{J} \subseteq S$ which is generated by a finite number of homogeneous elements of degree 1 (i.e. elements in S_1). We call $\mathcal{J}$ a *reduction* of S_+ with respect to M if $\left(\mathcal{J}M\right)_n = M_n$ or $\mathcal{J}_1 M_{n-1} = M_n$ for all large $n.$ The least integer n such that $\mathcal{J}_1 M_k = M_{k+1}$ for all $k \geq n$ is called the *reduction number* of S_+ with respect to $\mathcal{J}$ and $M,$ denote this number by $r_{\mathcal{J}}(S_+, M).$ The homogeneous ideal $\mathcal{J}$ is called a *minimal reduction* of S_+ with respect to M if $\mathcal{J}$ does not properly contain any other reductions of S_+ with respect to $M.$ The reduction number of S_+ with respect to $\mathcal{J}$ and M is defined by

$r(S_+, M) = \min\{r_{\mathcal{J}}(S_+, M) \mid \mathcal{J}$ is a minimal reduction of S_+ with respect to M $\}$ [18].

The relationship between the two concepts of reductions and filter-regular sequences is shown by the following proposition.

Proposition 2.4 [9]**.** *Let* $x_1,\ldots,x_t \in S_1$ *be a maximal* S_+ *-filter-regular sequence. Then* $x_1,\ldots,x_t$ *generate a reduction of* S_+ *with respect to M.*

The following lemma showed the relationship between the two concepts of filter-regular sequences and minimal reductions in the case that A is an Artinian ring.

Lemma 2.5 [19]**.** *Let* A *be an Artin ring. Assume that* $S_1 \not\subset \sqrt{\mathrm{Ann}_S(M)}.$ *Set* $\ell = \dim M.$ *Then each minimal reduction of* S_+ *with respect to M is generated by an* S_+ *-filter-regular sequence with respect to M consisting of* ℓ *homogeneous elements of degree 1.*

2.2. On the multiplicity of graded modules

Let (A, m) be a Noetherian local ring with maximal ideal $\mathrm{m},$ $M = \underset{n\geq 0}{\oplus} M_n$ a finitely generated graded S -module with $\dim M = \ell \geq 1$.

Remark 2.6. Since S is a finitely generated standard graded algebra over Noetherian ring A and M is a Noetherian S-module, there exists n_0 such that $S_{n-k}M_k = S_1^{n-k}M_k = M_n$ for all $n \geq n_0$. Moreover, we have $M_{n+k} = S_1^k M_n$ for all $n \geq n_0$ and $k \geq 0$.

Suppose that A is Artinian. By [20, Theorem 4.1], there exists a numerical polynomial $P(n)$ of degree $\ell - 1$ such that $\ell_A(M_n) = P(n)$ for large n. Set

$$P(n) = \frac{e(M)}{(\ell-1)!}n^{\ell-1} + f(n), \deg f(n) < \ell - 1.$$

Then $e(M)$ is a positive integer called the *multiplicity of graded module* M.

Remark 2.7. (i) We have $\dim M = \ell \geq 1$ if and only if $\ell_A(M_n) = P(n)$ is a polynomial of degree $\ell - 1 \geq 0$ for all large n. By Remark 2.6, this fact is equivalent to $M_{n+k} = S_1^k M_n \neq 0$ for all large n and for any $k \geq 0$. Therefore $\dim M = \ell \geq 1$ is equivalent to $S_1 \not\subset \sqrt{\operatorname{Ann}_S(M)}$.

(ii) By (i) and Proposition 2.3, there exists $x \in S_1$ such that x is an S_+ -filter-regular element with respect to M. Consider the surjective map $\lambda_x : M_n \to xM_n$, $z \mapsto xz$. We have $\operatorname{Ker}\lambda_x = (0_M : x) \cap M_n = (0_M : x)_n = 0$ for all large n by Remark 2.2(ii). Thus λ_x is isomorphic and $M_n \cong xM_n$ for all large n. We have

$$\ell_A\left(\left[\frac{M}{xM}\right]_n\right) = \ell_A\left(\frac{M_n}{xM_{n-1}}\right) = \ell_A(M_n) - \ell_A(xM_{n-1}) = \ell_A(M_n) - \ell_A(M_{n-1}) = P(n) - P(n-1)$$

for all large n. From these facts, we get $\dim \frac{M}{xM} = \dim M - 1 = \ell - 1$, $e\left(\frac{M}{xM}\right) = e(M)$.

(iii) By (i) and (ii), by induction on $\ell = \dim M$ for each maximal S_+ -filter-regular sequence $x_1, \ldots, x_t \in S_1$ with respect to M we have $t = \ell$ and for any $1 \leq i \leq \ell$, we get

$$\dim \frac{M}{(x_1, \ldots, x_i)M} = \dim M - i = \ell - i, \; e\left(\frac{M}{(x_1, \ldots, x_i)M}\right) = e(M).$$

2.3. On the multiplicity of fiber cones of graded modules in the case of arbitrary dimensions

Suppose that $S_1 \not\subset \sqrt{\operatorname{Ann}_S(M)}$. Set $F_J(S)_+ = \bigoplus_{n>0} \frac{S_n}{JS_n}$, $\ell = \dim F_J(M)$.

Then $F_J(M)$ is a finitely generated graded $F_J(S)$ -module and $F_J(S)$ is a finitely generated standard graded algebra over Artinian local ring A/J. By [20, Theorem 4.1], there exists a numerical polynomial $P(n)$ of degree $\ell - 1$ such that $\ell_A([F_J(M)]_n) = \ell_A\left(\frac{M_n}{JM_n}\right) = P(n)$ for all large n, where the degree of the polynomial 0 is -1.

Remark 2.8. (i) Since $\sqrt{J}=m$ and $F_J(S)=\dfrac{S}{JS}$, $F_J(M)=\dfrac{M}{JM}$ it follows that $\dim F_J(M)=\dim F_m(M)$.

(ii) Since $S_1 \not\subset \sqrt{\mathrm{Ann}_S(M)}$ we have $\ell \geq 1$. Indeed, if $\ell=0$ then $P(n)=\ell_A\left(\dfrac{M_n}{JM_n}\right)=0$ for all large n. Therefore $\dfrac{M_n}{JM_n}=0$ for all large n. By Nakayama's lemma, $M_n=0$ for all large n. From these facts and by Remark 2.6, there exists a positive integer n_0 such that $S_1^k M_n=M_{n+k}=0$ for all $n\geq n_0$ and for any $k\geq 0$. For $n\leq n_0-1$, we have $S_1^k M_n \subset M_{n+k}=0$ for all large *k*. Thus $S_1^k M_n=0$ for all *n* and for all large *k*. Hence, $S_1^k \subset \mathrm{Ann}_S(M)$ for all large *k* or $S_1 \subset \sqrt{\mathrm{Ann}_S(M)}$. This is contrary to the assumption.

By Remark 2.8(ii), $\deg P(n)=\ell-1\geq 0$. Write $P(n)$ in the form

$$P(n)=\frac{e\big(F_J(M)\big)}{(\ell-1)!}n^{\ell-1}+f(n),\quad \deg f(n)<\ell-1.$$

Then $e\big(F_J(M)\big)$ is a positive integer called the *multiplicity* of the fiber cone $F_J(M)$.

Remark 2.9. By Remark 2.7(ii), any maximal $F_J(S)_+$-filter-regular sequence with respect to $F_J(M)$ consists of ℓ elements. Assume that $x_1,\dots,x_\ell \in S_1$ such that $\overline{x_1},\dots,\overline{x_\ell}$ is a maximal $F_J(S)_+$-filter-regular sequence with respect to $F_J(M)$, where $\overline{x_1},\dots,\overline{x_\ell}$ are the images of $x_1,\dots,x_\ell$ in $\dfrac{S_1}{JS_1}$, respectively. By Remark 2.7(iii), we have

$$\dim\frac{F_J(M)}{(\overline{x_1},\dots,\overline{x_{\ell-1}})F_J(M)}=\ell-(\ell-1)=1,\quad e\left(\frac{F_J(M)}{(\overline{x_1},\dots,\overline{x_{\ell-1}})F_J(M)}\right)=e\big(F_J(M)\big).$$

On the other hand, we also have

$$\frac{F_J(M)}{(\overline{x_1},\dots,\overline{x_{\ell-1}})F_J(M)}=\bigoplus_{n\geq 0}\frac{M_n}{(x_1,\dots,x_{\ell-1})M_{n-1}+JM_n}=\bigoplus_{n\geq 0}\frac{M_n\big/(x_1,\dots,x_{\ell-1})M_{n-1}}{\big((x_1,\dots,x_{\ell-1})M_{n-1}+JM_n\big)\big/(x_1,\dots,x_{\ell-1})M_{n-1}}$$

$$=\bigoplus_{n\geq 0}\frac{M_n\big/(x_1,\dots,x_{\ell-1})M_{n-1}}{J\left(M_n\big/(x_1,\dots,x_{\ell-1})M_{n-1}\right)}=F_J\left(\frac{M}{(x_1,\dots,x_{\ell-1})M}\right).$$

Hence, we get $\dim F_J\left(\dfrac{M}{(x_1,\dots,x_{\ell-1})M}\right)=1$ and $e\big(F_J(M)\big)=e\left(F_J\left(\dfrac{M}{(x_1,\dots,x_{\ell-1})M}\right)\right)$.

The following theorem express the multiplicity formulas of graded fiber cones in terms of lengths of A-modules.

Theorem 2.10. *Let* $x_1,\ldots,x_\ell \in S_1$ *such that* $\overline{x_1},\ldots,\overline{x_\ell}$ *is a maximal* $F_J(S)_+$*-filter-regular sequence with respect to* $F_J(M)$, *where* $\overline{x_1},\ldots,\overline{x_\ell}$ *are the images of* $x_1,\ldots,x_\ell$ *in* $\dfrac{S_1}{JS_1}$, *respectively. Set* $\overline{M} = \dfrac{M}{(x_1,\ldots,x_{\ell-1})M}$, $r = r\left(F_J(S)_+, F_J(\overline{M})\right)$ *the reduction number of* $F_J(S)_+$ *with respect to* $F_J(\overline{M})$. *Then the following statements hold.*

(i) $e\left(F_J(M)\right) = \ell_A\left(\dfrac{\overline{M}_r}{\left(J\overline{M} : S_+^\infty\right)_r}\right)$.

(ii) $e\left(F_J(M)\right) = \ell_A\left(\dfrac{M_r}{\left([JM + (x_1,\ldots,x_{\ell-1})M] : S_+^\infty\right)_r}\right)$.

Proof. By Remar 2.9, we have $e\left(F_J(M)\right) = e\left(F_J(\overline{M})\right)$ and $\dim F_J(\overline{M}) = 1$.

(i) By [9, Theorem 2.9], $e\left(F_J(\overline{M})\right) = \ell_A\left(\dfrac{\overline{M}_r}{\left(J\overline{M} : S_+^\infty\right)_r}\right)$. Hence we get

$$e\left(F_J(M)\right) = \ell_A\left(\frac{\overline{M}_r}{\left(J\overline{M} : S_+^\infty\right)_r}\right).$$

(ii) We have

$$\overline{M}_r = \frac{M_r}{(x_1,\ldots,x_{\ell-1})M_{r-1}},\ \left(J\overline{M} : S_+^\infty\right)_r = \frac{[JM + (x_1,\ldots,x_{\ell-1})M] : S_+^\infty}{(x_1,\ldots,x_{\ell-1})M} = \frac{\left([JM + (x_1,\ldots,x_{\ell-1})M] : S_+^\infty\right)_r}{(x_1,\ldots,x_{\ell-1})M_{r-1}}.$$

Therefore $\dfrac{\overline{M}_r}{\left(J\overline{M} : S_+^\infty\right)_r} = \dfrac{M_r}{\left([JM + (x_1,\ldots,x_{\ell-1})M] : S_+^\infty\right)_r}$. Hence

$$e\left(F_J(M)\right) = \ell_A\left(\frac{\overline{M}_r}{\left(J\overline{M} : S_+^\infty\right)_r}\right) = \ell_A\left(\frac{M_r}{\left([JM + (x_1,\ldots,x_{\ell-1})M] : S_+^\infty\right)_r}\right).$$

In the case that $M = S$, set $\overline{S} = \dfrac{S}{(x_1,\ldots,x_{\ell-1})}$, $F_J(\overline{S}) = \bigoplus\limits_{n\geq 0}\dfrac{\overline{S}_n}{J\overline{S}_n}$, $F_J(\overline{S})_+ = \bigoplus\limits_{n>0}\dfrac{\overline{S}_n}{J\overline{S}_n}$ and $r = r\left(F_J(\overline{S})_+\right)$ the reduction number $F_J(\overline{S})_+$. It is clear that $r = r\left(F_J(S)_+, F_J(\overline{S})\right)$. By Theorem 2.10, we obtain the multiplicity formulas of fiber cones rings as follows.

Corollary 2.11. *Let* $x_1,\ldots,x_\ell \in S_1$ *such that* $\overline{x_1},\ldots,\overline{x_\ell}$ *is a maximal* $F_J(S)_+$*-filter-regular* *sequence with respect to* $F_J(S)$, *where* $\overline{x_1},\ldots,\overline{x_\ell}$ *are the images of* $x_1,\ldots,x_\ell$ *in* $\dfrac{S_1}{JS_1}$, *respectively. Set* $\overline{S}=\dfrac{S}{(x_1,\ldots,x_{\ell-1})}$, $r=r\left(F_J(\overline{S})_+\right)$ *the reduction number* $F_J(\overline{S})_+$. *Then the following statements hold.*

(i) $e\left(F_J(S)\right)=\ell_A\left(\dfrac{\overline{S}_r}{\left(J\overline{S}:S_+^\infty\right)_r}\right)$.

(ii) $e\left(F_J(S)\right)=\ell_A\left(\dfrac{S_r}{\left([JS+(x_1,\ldots,x_{\ell-1})]:S_+^\infty\right)_r}\right)$.

3. Conclusions

Using the concept of filter-regular sequences in graded modules, the paper has established the formula for calculating the multiplicity of graded fiber cone modules with arbitrary Krull dimensions in term of the lengths of modules. The paper's approach is based on the properties of the Hilbert function, the Hilbert polynomial with the difference formula and the mathematical induction method to transfer the calculation of the multiplicity of the general graded fiber cone modules to the case of the fiber cone modules with dimensions 1. The results obtained allow us to continue studying to establish the multiplicity formula for the fiber cone rings and the fiber cone modules in more specific cases. Up to now, the calculation of the multiplicity of the fiber cone rings and the fiber cone modules has been studied by many authors and approached with many different tools. However, solving the problem through filter-regular sequences is an approach that contributes to showing the role of special element sequences in the study of the multiplicity of rings and modules. It can be verified to see that each weak-(FC)-sequence of graded modules generates a filter-regular sequence of the corresponding fiber cone modules, so Theorem 2.10 still holds for the case of weak-(FC)-sequences.

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