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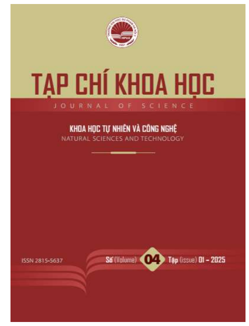

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The existence and uniqueness of weak solutions to three-dimensional Kelvin-Voigt equations with damping and unbounded delays

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Abstract

There are many results involving PDEs in fluid mechanics with delays and many results about asymptotic behavior to PDEs. Navier-Stokes equations with delays have been studied extensively over the last decades, for their important contributions to understanding fluid motion and turbulence. In this paper we consider the modifications of the three dimensional Navier-Stokes equations: the three dimensional Kelvin-Voigt equations involving damping and unbounded delays in a bounded domain $\Omega \subset \mathbb{R}^3$. The damping term is often introduced to model energy dissipation, which can stabilize the system. We show the existence and uniqueness of weak solutions by the Galerkin approximations method and the energy method.

Keywords: 3D Kelvin-Voigt equations, damping, delays, weak solutions, a Galerkin scheme

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. In this paper, we consider the following 3D Kelvin-Voigt equations with damping and delays in Ω,

$$\begin{cases} \partial_t(u-\alpha^2\Delta u) - \nu\Delta u + \nabla p + \kappa|u|^{\beta-1}u = g(t,u_t) + h(t) & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega, \\ u(x,t) = 0 & \text{in } (0,T)\times\partial\Omega, \\ u(\theta,x) = \phi(\theta,x), & \theta\in(-\infty,0], x\in\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, $\alpha > 0, \kappa > 0, \beta \geq 1$ are three constants, $u = u(x,t) = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, h is a nondelayed external force field, g

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is another external force term and contains hereditary characteristic u_t, where u_t is the function defined on $(-\infty, 0]$ by $u_t(\theta) = u(t+\theta)$, $\theta \in (-\infty, 0]$, u_0 is the initial velocity and ϕ the initial datum on the interval.

The case $\alpha \equiv 0$ and $g \equiv 0$ has been studied in [1] by X. Cai and Q. Jiu, the equation (1) becomes Navier-Stokes equation with damping.

Note that the case $\kappa \equiv 0$ and $g \equiv 0$ corresponds to the classical Navier-Stokes-Voigt problem. The existence, long-time behavior and regularity of solutions to the 3D Navier-Stokes-Voigt equations without delays in bounded domains and unbounded domains satisfying the Poincaré inequality have been studied by many mathematicians [2]–[11]. There are many results involving PDEs in fluid mechanics with delays [12]-[17]. However, all the results with finite delay (constant delay, bounded variable delay or bounded distributed delay) has been studied in the phase spaces $C([-h, 0); X)$ and $L^2(-h, 0; X)$, with suitable Banach space X, or infinite distributed delay in $C_\gamma(X)$, where

$$C_\gamma(X) = \left\{\varphi \in C((-\infty, 0]; X) \colon \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\right\} \ (\gamma > 0),$$

is the Banach space endowed with the norm

$$\| \varphi \|_\gamma = \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \| \varphi(\theta) \|_X.$$

In this paper, following the recent work [15] we continue studying the system (1) with unbounded variable delays in the following space

$$BCL_{-\infty}(X) = \left\{\varphi \in C((-\infty, 0]; X) \colon \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } X\right\}$$

which is a Banach space equipped with the norm

$$\| \varphi \|_{BCL_{-\infty}(X)} = \sup_{\theta \in (-\infty, 0]} \| \varphi(\theta) \|_X.$$

We will discuss the existence, uniqueness of weak solutions. The existence and uniqueness of the solution is proved by the classic Galerkin approximation and the energy method.

The rest of the paper is organized as follows. In section 2, we will set up some spaces and lemmas which will be used in the later sections. Section 3 will be devoted to the existence and uniqueness of solutions of the model.

2. Preliminaries

We consider the following space:

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 \colon \text{div}u = 0\}.$$

Let H be the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$ defined by

$$(u, v) = \sum_{j=1}^3 \int_\Omega u_j(x)v_j(x)dx \text{ for } u, v \in (L^2(\Omega))^3.$$

We also denote V, the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$, and associated scalar product $((\cdot,\cdot))$ defined by

$$((u, v)) = \sum_{i,j=1}^3 \int_\Omega \frac{\partial u_j}{\partial x_j}\frac{\partial v_j}{\partial x_i} dx \text{ for } u, v \in (H_0^1(\Omega))^3.$$

We use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle_{V,V'}$ for the dual pairing between V and V'. We recall the Stokes operator $A: V \to V'$ by $\langle Au, v\rangle = ((u, v))$. Denote by P the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V. Then $Au = -P\Delta u$, for all $u \in D(A) = (H^2(\Omega))^3 \cap V$. The

Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{\infty} \subset H$ such that $Aw_j = \lambda_j w_j$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \to +\infty \text{ as } t \to \infty.$$

We have the following Poincaré inequalities

$$\| u \|^2 \geq \lambda_1 |u|^2, \ \forall u \in V, \tag{2}$$

$$|u|^2 \geq \lambda_1 \| u \|_*^2, \ \forall u \in H.$$

From (2), we have

$$|u|^2 \geq d_0(|u|^2 + \alpha^2 \| u \|^2), \ \forall u \in V,$$

where $d_0 = \frac{\lambda_1}{1+\alpha^2\lambda_1}$. Furthermore, for $\alpha > 0$, the operator $I + \alpha^2 A$ has compact inverse $(I + \alpha^2 A)^{-1}: D(A)' \to H$ with the following estimate

$$\| (I + \alpha^2 A)^{-1} u \| \leq \alpha^{-2} \| u \|_*, \ \forall u \in V'.$$

We define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u, v, w \in V,$$

and $B: V \times V \to V'$ by $\langle B(u, v), w\rangle = b(u, v, w)$. We can write $B(u, v) = P[(u \cdot \nabla)v]$. It is easy to check that if $u, v, w \in V$, then $b(u, v, w) = -b(u, w, v)$, and in particular,

$$b(u, v, v) = 0, \quad \forall u, v \in V. \tag{3}$$

Using Hölder's inequality, Ladyzhenskaya's inequality, we can choose the best positive constant c_0 such that

$$|b(u, v, w)| \leq c_0 \| u \| \| v \| \, |w|^{1/2} \| w \|^{1/2}, \quad \forall u, v, w \in V. \tag{4}$$

From (4) and using Poincaré's inequality (2), we obtain that

$$|b(u, v, w)| \leq c_0 \lambda_1^{-1/4} \| u \| \| v \| \| w \|, \quad \forall u, v, w \in V. \tag{5}$$

We will assume that $f \in L^2(0, T; V')$. For the term g, we assume that $g: [0, T] \times BCL_{-\infty}(H) \to (L^2(\Omega))^3$, then

(g1) For any $\xi \in BCL_{-\infty}(H)$, the mapping $[0, T] \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^3$ is measurable.

(g2) $g(\cdot, 0) = 0$.

(g3) There exists a constant $L_g > 0$ such that, for any $t \in [0, T]$ and all $\xi, \eta \in BCL_{-\infty}(H)$,

$$|g(t, \xi) - g(t, \eta)| \leq L_g \| \xi - \eta \|_{BCL_{-\infty}(H)}.$$

Some examples of g which satisfier (g1) - (g3) can be seen in [18] for more details.

We can rewrite the 3D Kelvin-Voigt equations (1.1) in the following functional form

$$\begin{cases} \frac{\boldsymbol{d}}{\boldsymbol{dt}}(u + \alpha^2 Au) + \nu Au + B(u, u) + \kappa |u|^{\beta-1} u = Pg(t, u_t) + Ph(t), & \text{in } (0, T) \times \Omega, \\ u(\theta) = \phi(\theta), & \theta \in (-\infty, 0]. \end{cases} \tag{6}$$

3. The existence and uniqueness of weak solutions

We first give the definition of a weak solution.

Definition *Given an initial datum $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, a weak solution u to (1) in the interval $(-\infty, T]$, $T > 0$, is a function $u \in C((-\infty, T]; H) \cap C(0, T; V) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$ with $u(\theta) = \phi(\theta)$, $\theta \leq 0$ and $\frac{du}{dt} \in L^2(0, T; V) + L^{(\beta+1)/\beta}(0, T; L^{(\beta+1)/\beta}(\Omega))$ such that, for all $v \in V$, and a.e. $t \in (0, T)$*

$$\begin{aligned}\frac{d}{dt}((u(t), v) + \alpha^2((u(t), v))) + \nu((u(t), v)) \quad &+ b(u(t), u(t), v) + \langle \kappa |u|^{\beta-1} u, v\rangle \\ &= \langle h(t), v\rangle + (g(t, u_t), v).\end{aligned}$$

Now we show the existence and uniqueness of weak solutions.

Theorem *Consider $h \in L^2(0, T; V')$, $g: [0, T] \times BCL_{-\infty}(H) \to H$ satisfying (g1)-(g3) and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ are given. Then there exists a unique weak solution to (1).*

Proof. **(i) Existence.** We split the proof of the existence into several steps.

Step 1: A Galerkin scheme. Let $\{v_j\}_{j=1}^{\infty}$ be the basis consisting of eigenfunctions of the Stokes operator A, which is orthonormal in H and orthogonal in V. Denote $V_m = \text{span}\{v_1, \dots, v_m\}$ and consider the projector $P_m u = \sum_{j=1}^{m} (u, v_j) v_j$. Define also

$$u^m(t) = \sum_{j=1}^{m} \gamma_{m,j}(t) v_j,$$

where the coefficients $\gamma_{m,j}$ are required to satisfy the following system

$$\begin{aligned}\frac{d}{dt}((u^m(t), v_j) + \alpha^2((u^m(t), v_j)) \quad &+ \nu((u^m(t), v_j)) + b(u^m(t), u^m(t), v_j) \\ &+ \langle \kappa |u^m|^{\beta-1} u^m(t), v_j\rangle = \langle h(t), v_j\rangle + (g(u_t^m), v_j),\end{aligned} \tag{7}$$

for $j = 1, \dots, m$, and the initial condition $u^m(\theta) = P_m \phi(\theta)$ for $\theta \in (-\infty, 0]$.

The above system of ordinary functional differential equations with infinite delay in the unknown $(\gamma_{m,1}(t), \dots, \gamma_{m,m}(t))$ fulfills the conditions for the existence and uniqueness of local solutions (see [19], [20]). Hence, we conclude that the approximate solutions u^m to (7) exist unique locally on $[0, t^*)$ with $0 \leq t^* \leq T$. Next, we will obtain a priori estimates and ensure that the solutions u^m exist in $[0, T]$.

Step 2: A priori estimates. Multiplying (7) by $\gamma_{m,j}(t), j = 1, \dots, m$, summing up and using (3) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|u^m(t)|^2 + \alpha^2 \| u^m(t) \|^2) + \nu \| u^m(t) \|^2 \quad &+ \kappa \int_\Omega |u^m(t)|^{\beta+1} dx \\ &= \langle h(t), u^m(t)\rangle + (g(u_t^m), u^m(t)).\end{aligned}$$

Using the Cauchy inequality and noting that $|u^m(t)| \leq \| u_t^m \|_{BCL_{-\infty}(H)}$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|u^m(t)|^2 + \alpha^2 \| u^m(t) \|^2) + \nu \| u^m(t) \|^2 + \kappa \int_\Omega |u^m|^{\beta+1} dx \\ &\leq \| h(t) \|_* \| u^m(t) \| + L_g \| u_t^m \|_{BCL_{-\infty}(H)} |u^m(t)| \\ &\leq \frac{\nu}{2} \| u^m(t) \|^2 + \frac{\| h(t) \|_*^2}{2\nu} + L_g \| u_t^m \|_{BCL_{-\infty}(H)}^2,\end{aligned}$$

and hence

$$\begin{aligned}&\frac{d}{dt}(|u^m(t)|^2 + \alpha^2 \| u^m(t) \|^2) + \nu \| u^m(t) \|^2 + 2\kappa \int_\Omega |u^m|^{\beta+1} dx \\ \leq \quad &\frac{\| h(t) \|_*^2}{\nu} + 2L_g \| u_t^m \|_{BCL_{-\infty}(H)}^2.\end{aligned}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}&|u^m(t)|^2 + \alpha^2 \| u^m(t) \|^2 + \nu \int_0^t \| u^m(s) \|^2 ds + 2\kappa \int_0^t \int_\Omega |u^m|^{\beta+1} dx ds \\ \leq \quad &|u^m(0)|^2 + \alpha^2 \| u^m(0) \|^2 + \frac{1}{\nu} \int_0^t \| h(s) \|_*^2 ds + 2L_g \int_0^t \| u_s^m \|_{BC\ -\infty(H)}^2 ds.\end{aligned} \tag{8}$$

In particular, for any $t > 0$

$$\sup_{-t<\theta\leq 0}|u^m(t+\theta)|^2 + \alpha^2 \parallel u^m(t) \parallel^2 \leq \parallel \phi \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel \phi(0) \parallel^2$$
$$+\frac{1}{\nu}\int_0^t \parallel h(s) \parallel_*^2 ds + 2L_g \int_0^t (\parallel u_s^m \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel u^m(s) \parallel^2) ds.$$

Since

$$\begin{aligned} &\parallel u_t^m \parallel^2_{BC_{-\infty}(H)} + \alpha^2 \parallel u^m(t) \parallel^2 \\ = \;& \max\{\sup_{-t<\theta\leq 0}|u^m(t+\theta)|^2 + \alpha^2 \parallel u^m(t) \parallel^2; \sup_{\theta\leq -t}|u^m(t+\theta)|^2 + \alpha^2 \parallel u^m(t) \parallel^2\} \\ \leq \;& \max\{\sup_{-t<\theta\leq 0}|u^m(t+\theta)|^2 + \alpha^2 \parallel u^m(t) \parallel^2; \parallel \phi \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel u^m(t) \parallel^2\}, \end{aligned}$$

we obtain

$$\begin{aligned} \parallel u_t^m \parallel^2_{BC_{-\infty}(H)} &+\alpha^2 \parallel u^m(t) \parallel^2 \leq 2 \parallel \phi \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel \phi(0) \parallel^2 \\ &+\tfrac{1}{\nu}\textstyle\int_0^t \parallel h(s) \parallel_*^2 ds + 2L_g \int_0^t (\parallel u_s^m \parallel^2_{BC_{-\infty}(H)} + \alpha^2 \parallel u^m(s) \parallel^2) ds. \end{aligned}$$

By the Gronwall inequality we have

$$\begin{aligned} &\parallel u_t^m \parallel^2_{BCL_{-\infty}H)} + \alpha^2 \parallel u^m(t) \parallel^2 \\ &\leq e^{2L_g t}\left(\parallel \phi \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel \phi(0) \parallel^2 + \tfrac{1}{\nu}\textstyle\int_0^t (\parallel h(s) \parallel_*^2) ds\right). \end{aligned}$$

Then we obtain the following estimate: for any $R > 0$ such that $\parallel \phi \parallel_{BC_{-\infty}(H)} \leq R$, there exists a constant C depending on ν, L_g, f, such that

$$\parallel u_t^m \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel u^m(t) \parallel^2 \leq C(T,R), \forall t \in [0,T], \ \forall m \geq 1. \tag{9}$$

In particular,

$\{u^m\}$ is uniformly bounded in $L^\infty(0,T; BCL_{-\infty}(H)) \cap L^\infty(0,T;V)$.

From (8) and the above uniform estimates, we obtain

$$\begin{aligned} &\nu \textstyle\int_0^t \parallel u^m(s) \parallel^2 ds + 2\kappa \int_0^t \int_\Omega |u^m(s)|^{\beta+1} dx ds \\ \leq \;& |u^m(0)|^2 + \alpha^2 \parallel u^m(0) \parallel^2 + \tfrac{1}{\nu}\textstyle\int_0^t \parallel h(s) \parallel_*^2 ds + 2L_g \int_0^t \parallel u_s^m \parallel^2_{BC_{-\infty}(H)} ds \\ \leq \;& |u^m(0)|^2 + \alpha^2 \parallel u^m(0) \parallel^2 + \textstyle\int_0^t \left(\tfrac{1}{\nu} \parallel h(s) \parallel_*^2 + 2L_g C(T,R)\right) ds. \end{aligned}$$

Then we can conclude that $\{u^m\}$ is uniformly bounded in $L^2(0,T;V) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$.

Now, we prove the boundedness of $\{\frac{du^m}{dt}\}$. We have

$$\begin{aligned} \frac{d}{dt}(u^m(t) + \alpha^2 A u^m(t)) = \;& -\nu A u^m(t) - P_m B(u^m, u^m) - \kappa |u^m|^{\beta-1} u^m \\ & +P_m h(t) + P_m g(t, u_t^m). \end{aligned} \tag{10}$$

From (5), (9) and (10), we obtain

$$\begin{aligned} &\parallel \tfrac{d}{dt}(u^m + \alpha^2 A u^m) \parallel_* \\ \leq \;& \nu \parallel A u^m \parallel_* + \parallel B(u^m, u^m) \parallel_* + \kappa \parallel u^m \parallel_{(L^{\beta+1})^*} + \parallel h(t) \parallel_* + \parallel g(t, u_t^m) \parallel_* \\ \leq \;& \nu \parallel u^m \parallel + c_0 \lambda_1^{-1/4} \parallel u^m \parallel + \kappa \parallel u^m \parallel_{L^{(\beta+1)/\beta}} + \parallel h(t) \parallel_* + \lambda_1^{-1/2} |g(t, u_t^m)| \\ \leq \;& \nu \parallel u^m \parallel + c_0 \lambda_1^{-1/4} \parallel u^m \parallel + \kappa \parallel u^m \parallel_{L^{(\beta+1)/\beta}} + \parallel h(t) \parallel_* + L_g \lambda_1^{-1/2} \parallel u^m \parallel_{BCL_{-\infty}(H)} \\ \leq \;& C(T,R), \ \forall m \geq 1. \end{aligned}$$

This implies that $\frac{d}{dt}(u^m + \alpha^2 A u^m)$ is uniformly bounded in

$$L^2(0,T;V)+L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega)).$$

Then $\{\frac{du^m}{dt}\}$ is uniformly bounded in $L^2(0,T;V)+L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$.

Step 3. Approximation in $BCL_{-\infty}(H)$ of the initial datum.

We will show that

$$P_m\phi \to \phi \text{ in } BCL_{-\infty}(H). \tag{11}$$

Assume the contrary that (11) is not true. Then there exists $\epsilon>0$ and a subsequence, relabeled the same, such that

$$\| P_m\phi(\theta_m)-\phi(\theta_m) \| > \epsilon,\ \forall m. \tag{12}$$

One can assume that $\theta_m\to-\infty$, otherwise if $\theta_m\to\theta$, then $P_m\phi(\theta_m)\to\phi(\theta)$, since $\| P_m\phi(\theta_m)-P_m\phi(\theta) \| \le \| P_m\phi(\theta_m)-P_m\phi(\theta) \| + \| P_m\phi(\theta)-\phi(\theta) \| \to 0$ as $m\to+\infty$. But with $\theta_m\to-\infty$ as $m\to+\infty$, if we denote $x=\lim_{\theta\to-\infty}\phi(\theta)$, we obtain that

$$\| P_m\phi(\theta_m)-\phi(\theta_m) \| = \| P_m\phi(\theta_m)-P_mx \| + \| P_mx-x \| + \| x-\phi(\theta_m) \| \to 0.$$

This contradicts (12), so (11) holds.

Step 4: Compactness results. We obtain

$$u^m \rightharpoonup^* u \text{ weakly in } L^\infty(0,T;V),$$

$$u^m \rightharpoonup u \text{ in } L^2(0,T;V)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)),$$

$$\frac{du^m}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;V)+L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega)).$$

Since $\{u^m\}$ is uniformly bounded in $L^2(0,T;V)$ and $\{\frac{du^m}{dt}\}$ is uniformly bounded in $L^2(0,T;V)+L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$ and by the Aubin-Lions compactness lemma, we deduce that $u^m\to u$ strongly in $L^2(0,T;(L^2(\Omega))^3)$. Thus, we have (up to a subsequence)

$$u^m \to u \text{ a.e. in } \Omega_T.$$

From $\{u^m\}$ is uniformly bounded in $L^2(0,T;V)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$ and $\{\frac{du^m}{dt}\}$ is uniformly bounded in $L^2(0,T;V)+L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$, by direct estimate (following the same technique as in [3]) we can show that $\{u^m\}$ is Cauchy sequence in $C([0,T];V)$. Thus,

$$u^m \to u \text{ in } C([0,T];V). \tag{13}$$

We have

$$\begin{aligned} &\sup_{\theta\le 0}|u^m(t+\theta)-u(t+\theta)| \\ \le\ &\max\{\sup_{\theta\le -t}|P_m\phi(\theta+t)-\phi(\theta+t)|, \sup_{-t\le\theta\le 0}|u^m(t+\theta)-u(t+\theta)|\} \\ \le\ &\max\{|P_m\phi-\phi|_{BCL_{-\infty}(H)}, \sup_{-t\le\theta\le 0}|u^m(t+\theta)-u(t+\theta)|\}\to 0. \end{aligned}$$

Then (11) and (13) imply that

$$u_t^m \to u_t \text{ in } BCL_{-\infty}(H),\ \forall t\in[0,T].$$

Therefore, taking into account (g3), we have

$$g(\cdot,u_\cdot^m)\to g(\cdot,u_\cdot) \text{ in } L^2(0,T;H).$$

Finally, we can pass to the limit in (7) and conclude that u solves (1).

(ii) Uniqueness. Let u,v be two weak solutions of problem (6) with respect to the initial datum ϕ and ψ in $BCL_{-\infty}(H)$ with $\phi(0),\psi(0)$ in V.

Setting $w = u - v$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w|^2 \quad &+\alpha^2 \parallel w \parallel^2) + \nu \parallel w \parallel^2 + b(u,u,w) - b(v,v,w) \\ &+ \int_\Omega \kappa(|u|^{\beta-1}u - |v|^{\beta-1}v)(u-v)dx = (g(u_t) - g(v_t), w),\end{aligned} \tag{14}$$

and $w(\theta) = \phi(\theta) - \psi(\theta),\ \theta \in (-\infty, 0]$.

It is well known (see, e.g. [22]) that there exist two nonnegative constants $\mu = \mu(\beta, \kappa)$ such that

$$\int_\Omega (|u|^{\beta-1}u - |v|^{\beta-1}v)(u-v)dx \geq \mu \int_\Omega (|u|^{\beta-1} + |v|^{\beta-1})|u-v|^2 dx \geq 0. \tag{15}$$

Using (15), (14) becomes

$$\begin{aligned}&\frac{d}{dt}(|w|^2 + \alpha^2 \parallel w \parallel^2) + 2\nu \parallel w \parallel^2 + \mu \int_\Omega (|u|^{\beta-1} + |v|^{\beta-1})|u-v|^2 dx \\ \leq \quad & 2|b(w,u,w)| + 2\int_\Omega |g(u_t) - g(v_t)| \cdot |w(t)| dx.\end{aligned}$$

From (4) and the Young inequality we have

$$|b(w,u,w)| \leq c_0 \parallel u \parallel |w|^{1/2} \parallel w \parallel^{3/2} \leq \frac{\nu}{4} \parallel w \parallel^2 + \mathrm{C} \parallel u \parallel^4 |w|^2,$$

where $C = C(c_0, \nu)$. Taking (g3) into account and using Young's inequality, we get

$$\begin{aligned}\frac{d}{dt}(|w(t)|^2 + \alpha^2 \parallel w(t) \parallel^2) + 2\nu \parallel w \parallel^2 &\leq \frac{\nu}{2} \parallel w \parallel^2 + \mathrm{C} \parallel u \parallel^4 |w|^2 + 2L_g \parallel w_t \parallel_{BCL_{-\infty}(H)} |w(t)| \\ &\leq \frac{\nu}{2} \parallel w \parallel^2 + \mathrm{C} \parallel u \parallel^4 |w|^2 + \frac{\nu\lambda_1}{2}|w|^2 + \frac{L_g}{\lambda_1} \parallel w \parallel^2_{BCL_{-\infty}(H)} \\ &\leq \frac{\nu}{2} \parallel w \parallel^2 + \mathrm{C} \parallel u \parallel^4 |w|^2 + \frac{\nu}{2} \parallel w \parallel^2 + \frac{L_g}{\lambda_1} \parallel w \parallel^2_{BC\ \ _{-\infty}(H)}.\end{aligned}$$

Thus, we have

$$\begin{aligned}|w(t)|^2 + \alpha^2 \parallel w(t) \parallel^2 \leq |w(0)|^2 + \alpha^2 \parallel w(0) \parallel^2 + C \int_0^t \parallel u(s) \parallel^4 |w(s)|^2 ds \\ + \frac{L_g}{\lambda_1} \int_0^t \parallel w_s \parallel^2_{BCL_{-\infty}(H)} ds.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}\parallel w_t \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel w(t) \parallel^2 \leq \parallel \phi - \psi \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel \phi(0) - \psi(0) \parallel^2 \\ + \int_0^t \left(\mathrm{C} \parallel u(s) \parallel^4 + \frac{L_g}{\lambda_1}\right)\left(\parallel w_s \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel w(s) \parallel^2\right) ds\end{aligned}$$

Applying the Gronwall inequality on [0, T], we obtain that

$$\begin{aligned}&\parallel u_t - v_t \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel u(t) - v(t) \parallel^2 \\ &\leq (\parallel \phi - \psi \parallel^2_{BCL_{-\infty}(H)} + \alpha^2 \parallel \phi(0) - \psi(0) \parallel^2) \times exp\left(C \int_0^T \parallel u(s) \parallel^4 \mathrm{ds} + \frac{L_g}{\lambda_1} T\right).\end{aligned}$$

Since u in $L^\infty(0,T;V)$, we complete the proof of uniqueness.

4. Conclusion

In this paper, we have presented the 3D Kelvin-Voigt equations involving damping and unbounded delays in a bounded domain $\Omega \subset \mathbb{R}^3$. We have shown the existence and uniqueness of weak solutions by the Galerkin approximations method and the energy method. It is meaningful if we can establish the existence of stationary solutions and global attractors for this model. We aim to investigate these problems in future work.

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