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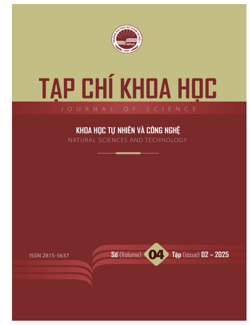

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An eigenvalue approach to the dynamics of supply-demand-price systems

Thanh-Huyen Pham Thi*, Anh-Thanh Le

Hanoi University of Industry, Hanoi, Vietnam

Abstract

This paper examines the oscillatory behavior of supply-demand-price dynamical systems using eigenvalue analysis. Unlike traditional stability assessments, our study reveals that the system does not exhibit asymptotic stability since all eigenvalues have zero real parts. This results in sustained harmonic oscillations rather than convergence to equilibrium. By formulating market dynamics through differential equations and analyzing the Jacobian matrix, we characterize the system's long-term behavior based on its eigenvalues. Our findings provide mathematical formulations, theoretical insights, and numerical simulations that illustrate persistent price fluctuations and cyclical market behavior. The study enhances the understanding of market instability, hence emphasizing the role of linear algebra in economic dynamics and its implications for economic modeling and policy-making.

Keywords: Eigenvalues, eigenvectors, stability analysis, market equilibrium, dynamical systems

1. Introduction

Understanding the stability of market dynamics is a fundamental issue in economic theory and policy-making. Classical economic models describe the interactions between supply, demand, and price through differential equations, reflecting how markets adjust over time (see, for instance, [1], [2] and [3]). The stability of these dynamical systems plays a crucial role in determining whether an economy converges to equilibrium or undergoes prolonged oscillations (see, for instance, [4]–[7]). Numerous studies, including those by Samuelson [7], Arrow and Debreu [6], and Hahn [5], have laid the theoretical foundation for equilibrium stability, while later works have extended these analyses using advanced mathematical tools (see, for instance, [8], [9] and [10]).

* Corresponding author, E-mail: huyenptt1@haui.edu.vn

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In this paper, we analyze the dynamics of the supply-demand-price system using the eigenvalues and eigenvectors of the system's Jacobian matrix. The results indicate that the system is unstable, as all eigenvalues have zero real parts, leading to harmonic oscillations without convergence to equilibrium. By representing the system in matrix form, we apply techniques from linear algebra and dynamical systems theory to characterize the oscillatory nature of the market (see, for instance, [11], [12] and [13]). This analysis clarifies the mechanisms that prevent prices, supply, and demand from reaching a stable state, instead causing them to fluctuate over time (see, for instance, [14], [15] and [3]). The use of eigenvalue-based criteria has been explored in mathematical economics and dynamical systems [16], [17], and our study extends these methods to explicitly describe the oscillatory process in market equilibrium. Additionally, the application of eigenvalue decomposition in mathematical modeling has been widely adopted in fields such as cryptography and information security, demonstrating the fundamental importance of matrix methods across diverse disciplines (see, for instance, [18]).

Our study builds upon classical works in economic dynamics, including Samuelson's stability analysis [7] and the equilibrium conditions established by Arrow and Debreu [6]. Furthermore, we incorporate mathematical techniques from differential equations and linear algebra (see, for instance, [17]–[20]) to describe the conditions under which the market exhibits harmonic oscillations rather than stability. Unlike prior research that primarily relies on numerical simulations, we focus on analytical conditions for the system's oscillatory behavior, thereby making our results more broadly applicable. Our approach is motivated by previous studies in economic dynamics, recursive macroeconomic theory, and the application of linear algebra in market analysis.

The main contributions of this paper are as follows:

- Formulating a dynamical system describing the evolution of supply, demand, and price, integrating standard economic adjustment equations into a unified mathematical framework.

- Identifying the conditions under which the dynamical system is unstable, leading to harmonic oscillations rather than convergence to equilibrium.

- Illustrating the oscillatory behavior of the market through a representative numerical simulation, without focusing on parameter sensitivity.

The remainder of this paper is organized as follows. Section 2 introduces the supply-demand-price dynamical system and its matrix representation. Section 3 presents the eigenvalue-based oscillation analysis, proving key theorems on harmonic oscillatory states. Section 4 applies these results to economic models and provides numerical examples that illustrate market oscillations. Finally, Section 5 concludes with implications for economic policy and future research directions.

2. Supply-Demand-Price Dynamical Systems Models

The supply, demand, and price adjustment equations appear in reference [3]. We have combined them into a dynamical system to analyze market evolution over time. By representing the system in matrix form, we can use eigenvalues and eigenvectors to examine the stability of the equilibrium state.

Throughout this paper, let $S(t)$, $D(t)$ and $P(t)$ denote the supply quantity, demand quantity, and price at time t, respectively.

The supply quantity increases or decreases depending on price changes and the deviation from the equilibrium level. The supply adjustment equation is given by:

$$\frac{dS(t)}{dt} = \alpha\left(P(t) - P^*\right), \quad \alpha > 0, \tag{1}$$

where α is the supply adjustment coefficient, and P^* is the equilibrium price.

The demand quantity varies inversely with price. When the price increases, demand decreases, and vice versa. The demand adjustment equation is given by:

$$\frac{dD(t)}{dt} = -\beta\left(P(t) - P^*\right), \quad \beta > 0, \tag{2}$$

where β is the demand adjustment coefficient.

The price changes depending on the discrepancy between demand and supply. The price adjustment equation is given by:

$$\frac{dP(t)}{dt} = \gamma\left(D(t) - S(t)\right), \quad \gamma > 0, \tag{3}$$

where γ is the price adjustment speed.

By combining equations (1), (2), and (3), we obtain the first-order differential equation system:

$$\begin{cases} \dfrac{dS(t)}{dt} & = \alpha\left(P(t) - P^*\right), \\ \dfrac{dD(t)}{dt} & = -\beta\left(P(t) - P^*\right), \\ \dfrac{dP(t)}{dt} & = \gamma\left(D(t) - S(t)\right). \end{cases} \tag{4}$$

Equilibrium occurs when $\frac{dS(t)}{dt} = \frac{dD(t)}{dt} = \frac{dP(t)}{dt} = 0$, leading to $S(t) = D(t)$ and $P(t) = P^*$.

Let $X(t) = \begin{bmatrix} S(t) - P^* \\ D(t) - P^* \\ P(t) - P^* \end{bmatrix}$, then the system (4) can be rewritten in matrix form as:

$$\frac{dX(t)}{dt} = AX(t)$$

where A is the Jacobian matrix at the equilibrium state.

The elements of matrix A are the partial derivatives of functions f_1, f_2, f_3 with respect to variables S, D, P, given by:

$$A = \begin{bmatrix} \dfrac{\partial f_1}{\partial S} & \dfrac{\partial f_1}{\partial D} & \dfrac{\partial f_1}{\partial P} \\ \dfrac{\partial f_2}{\partial S} & \dfrac{\partial f_2}{\partial D} & \dfrac{\partial f_2}{\partial P} \\ \dfrac{\partial f_3}{\partial S} & \dfrac{\partial f_3}{\partial D} & \dfrac{\partial f_3}{\partial P} \end{bmatrix}.$$

By direct computation, we obtain the Jacobian matrix as:

$$A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & -\beta \\ -\gamma & \gamma & 0 \end{bmatrix}.$$

The model in equation (4) (or its reduced form (5)) is used in analyzing factors that affect market stability, studying price fluctuations when supply and demand are imbalanced, and simulating markets under specific initial conditions. For better detailed explanations, it is essential to analyze the relationship between the eigenvalues of the dynamic system matrix and the behavior of the linear system.

3. Eigenvalue-Based Stability

The following theorem provides the general solution for linear differential systems. Although it can be derived from a more general case (see references [17]), we present the proof here for the reader's convenience.

Theorem 3.1. *Let A be a 3×3 square matrix with eigenvalues λ_1, λ_2, λ_3 and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Then, the general solution of the system*

$$\frac{dX(t)}{dt} = AX(t) \tag{5}$$

is given by

$$X(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3, \tag{6}$$

where c_1, c_2 and c_3 are arbitrary constants determined by the initial conditions.

Proof. We seek a solution of the form:

$$X(t) = e^{\lambda t}\mathbf{v},$$

where $\mathbf{v}$ is a vector and λ is an unknown real or complex number. Substituting into equation (5), we obtain:

$$\frac{d}{dt}\left(e^{\lambda t}\mathbf{v}\right) = A\left(e^{\lambda t}\mathbf{v}\right).$$

Since $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$, we get:

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v}.$$

Dividing both sides by $e^{\lambda t} \neq 0$, we have

$$A\mathbf{v} = \lambda\mathbf{v}.$$

This means that λ must be an eigenvalue of A and $\mathbf{v}$ is the corresponding eigenvector.

Now, assuming A has three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, these eigenvectors form a basis of $\mathbb{R}^3$, allowing the general solution to be written as:

$$X(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3,$$

where c_1, c_2, c_3 are arbitrary constants determined by the initial condition $X(0)$.

Differentiate both sides of the above equality with respect to t, we obtain

$$\frac{dX(t)}{dt} = c_1\lambda_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2\lambda_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3\lambda_3 e^{\lambda_3 t}\mathbf{v}_3. \tag{7}$$

On the other hand, since $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all $i = 1,2,3$, we have

$$AX(t) = A\left(c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3\right).$$

Using the properties of the matrix, we get

$$AX(t) = c_1 e^{\lambda_1 t} A\mathbf{v}_1 + c_2 e^{\lambda_2 t} A\mathbf{v}_2 + c_3 e^{\lambda_3 t} A\mathbf{v}_3.$$

Substituting $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ into the above equality, we obtain

$$AX(t) = c_1 e^{\lambda_1 t}\lambda_1\mathbf{v}_1 + c_2 e^{\lambda_2 t}\lambda_2\mathbf{v}_2 + c_3 e^{\lambda_3 t}\lambda_3\mathbf{v}_3. \tag{8}$$

From (7) and (8), it follows that

$$\frac{dX(t)}{dt} = AX(t).$$

This confirms that the solution of (5) is determined by (6), thereby further proving the theorem. □

In the analysis of market dynamics, an important state to consider is when supply, demand, and price remain unchanged over time. If the system reaches a point where these elements stay constant without oscillations or upward/downward trends, the market is said to have reached a stable equilibrium. This concept is formally defined as follows:

Definition 3.1. [cf. 15] A steady-state equilibrium in the supply-demand-price dynamical system is an equilibrium path in which

$$S(t) = S^*, \quad D(t) = D^*, \quad P(t) = P^* \quad \text{for all } t.$$

This means that the supply, demand, and price remain constant over time, indicating a stable market state.

The general solution in Theorem 3.1 shows that the system's behavior is determined by the eigenvalues of matrix A. If all eigenvalues have negative real parts, the solution $X(t)$ converges to zero as $t \to \infty$, ensuring stability. However, if any eigenvalue has a positive real part, some components of $X(t)$ will grow unbounded, making the system unstable.

This leads to the following key result on stability:

Theorem 3.2. *Let A be a 3×3 square matrix with eigenvalues λ_1, λ_2, λ_3 and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Then the linear differential system*

$$\frac{dX(t)}{dt} = AX(t).$$

is asymptotically stable if and only if all eigenvalues of A have negative real parts, i.e.,

$$\text{Re}(\lambda_1) < 0, \quad \text{Re}(\lambda_2) < 0, \quad \text{Re}(\lambda_3) < 0.$$

Proof.

Since A has three eigenvalues (not necessarily distinct) $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, it follows from Theorem 3.1 that the general solution of the system is given by

$$X(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3,$$

where c_1, c_2, c_3 are arbitrary constants determined by the initial condition.

Sufficient condition. We consider each case regarding the sign of the real part $\mathrm{Re}(\lambda_i)$. Suppose all eigenvalues of A have negative real parts, that is:

$$\mathrm{Re}(\lambda_1) < 0, \quad \mathrm{Re}(\lambda_2) < 0, \quad \mathrm{Re}(\lambda_3) < 0.$$

Consider each term in the general solution:

$$e^{\lambda_i t} = e^{(\mathrm{Re}(\lambda_i) + i\mathrm{Im}(\lambda_i))t} = e^{\mathrm{Re}(\lambda_i)t} e^{i\mathrm{Im}(\lambda_i)t}.$$

Since $\mathrm{Re}(\lambda_i) < 0$, we have $e^{\mathrm{Re}(\lambda_i)t} \to 0$ as $t \to \infty$. The term $e^{i\mathrm{Im}(\lambda_i)t}$ (which is a harmonic oscillation function) only induces oscillations without increasing the amplitude. It follows that $e^{\lambda_i t}\mathbf{v}_i \to 0$ as $t \to \infty$, leading to $\mathbf{x}(t) \to 0$. This proves that the system is stable.

Necessary condition. Suppose the system (5) is stable, that is

$$X(t) \to 0 \quad \text{khi} \quad t \to \infty.$$

This means that every term $e^{\lambda_i t}\mathbf{v}_i$ must tend to 0. Consider the following cases:

Case 1. If there exists λ_i with $\mathrm{Re}(\lambda_i) > 0$, then $e^{\lambda_i t}$ will grow unbounded as $t \to \infty$, causing $X(t)$ to also grow unbounded. This contradicts the assumption that the system is asymptotically stable. Therefore, it is not possible to have $\mathrm{Re}(\lambda_i) > 0$

Case 2. If there exists at least one λ_i with $\mathrm{Re}(\lambda_i) = 0$, meaning λ_i is purely imaginary, then $e^{\lambda_i t}$ does not tend to 0 but oscillates indefinitely. In this case, the solution $X(t)$ does not converge to 0, meaning the system may be stable but not asymptotically stable. This contradicts the assumption that the system is asymptotically stable. Therefore, the necessary condition for asymptotic stability is that all eigenvalues of A must have negative real parts.

We have shown that the dynamical system (5) is asymptotically stable if and only if all eigenvalues of A have negative real parts. This completes the proof of the theorem. □

4. Application to the Supply-Demand-Price System

In this section, we apply Theorems 3.1 and 3.2 to analyze the stability of the supply-demand-price system (4). Following the methodologies outlined in [18], [19], and [20], we compute the eigenvalues of the matrix A by solving the characteristic equation:

$$\det(A - \lambda I) = 0.$$

Expanding the determinant, we obtain

$$\begin{vmatrix} -\lambda & 0 & \alpha \\ 0 & -\lambda & -\beta \\ -\gamma & \gamma & -\lambda \end{vmatrix} = 0.$$

Using the determinant expansion technique from [18], we obtain the characteristic equation

$$\lambda(\lambda^2 + \gamma\alpha + \gamma\beta) = 0.$$

Solving for λ, we obtain the eigenvalues:

$$\lambda_1 = 0,\ \lambda_2 = \sqrt{-\gamma(\alpha+\beta)} \text{ and } \lambda_3 = -\sqrt{-\gamma(\alpha+\beta)}$$

Since α, β, and γ are given positive real numbers, all eigenvalues have zero real parts. Therefore, system (4) is not asymptotically stable; instead, it exhibits purely harmonic oscillations.

Example 4.1. In this example, we analyze the dynamic behavior of the supply-demand-price system governed by the differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} & = 1.2\big(P(t) - 50\big) \\ \dfrac{dD(t)}{dt} & = -1.0\big(P(t) - 50\big) \\ \dfrac{dP(t)}{dt} & = 2.0\big(D(t) - S(t)\big) \end{cases}$$

where

- Supply adjustment coefficient: $\alpha = 1.2$,
- Demand adjustment coefficient: $\beta = 1.0$,
- Price adjustment speed: $\gamma = 2.0$,
- Equilibrium price: $P^* = 50$.

Jacobian matrix:

$$A = \begin{bmatrix} 0 & 0 & 1.2 \\ 0 & 0 & -1.0 \\ -2.0 & 2.0 & 0 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = \sqrt{-4.4}, \quad \lambda_3 = -\sqrt{-4.4}$

Since the matrix has purely imaginary eigenvalues, the corresponding system will exhibit harmonic oscillations without damping or growth over time.

Below is the graph illustrating the solution of the supply-demand-price system over time with the initial conditions $S(0) = 20$, $D(0) = 15$ and $P(0) = 50$.

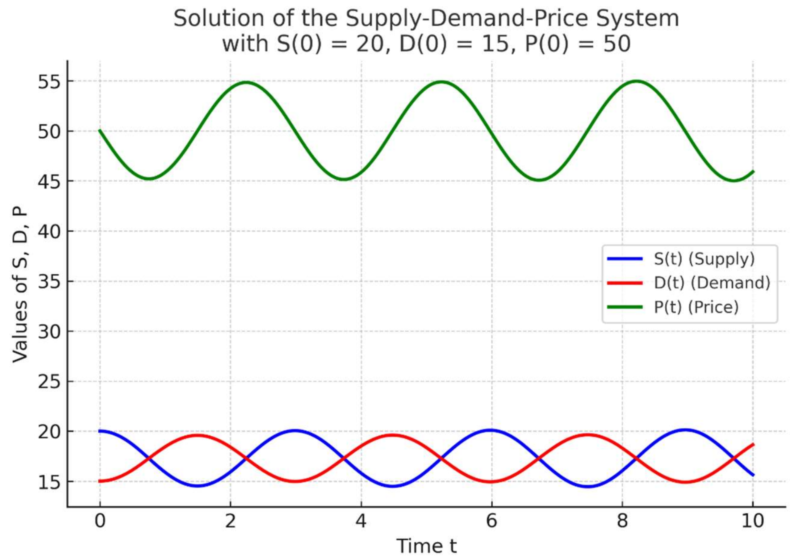

Figure 1. Supply–Demand–Price Oscillations

Since the eigenvalues are purely imaginary, the system exhibits undamped oscillations. To confirm this behavior, we solve the system numerically using the initial conditions $S(0)=20$, $D(0)=15$, and $P(0)=50$. The numerical solution, plotted in Figure 1, demonstrates periodic fluctuations in supply, demand, and price over time, corroborating the theoretical stability analysis.

These results indicate that the system does not converge to an equilibrium but instead sustains cyclical variations, which may be interpreted as persistent market oscillations. Such behavior suggests that external interventions, such as policy adjustments or demand regulation, might be necessary to stabilize the market.

5. Conclusion

In this paper, we analyzed the oscillatory behavior of the supply-demand-price system using eigenvalue analysis. The results indicate that the system is not asymptotically stable, as all eigenvalues have zero real parts, leading to sustained harmonic oscillations instead of convergence to equilibrium. By modeling the system through the Jacobian matrix, the role of linear algebra in forecasting market fluctuations is emphasized.

Our study suggests that price fluctuations can persist indefinitely in the absence of regulatory mechanisms. This highlights the potential importance of economic policies in stabilizing market dynamics. By introducing appropriate regulatory interventions, policymakers may reduce the amplitude or persistence of such oscillations, hence contributing to more stable economic environments.

These insights open up various opportunities for future research directions on the impact of policy design, external shocks, and nonlinear factors to enhance the practical applicability of the model in economic analysis.

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