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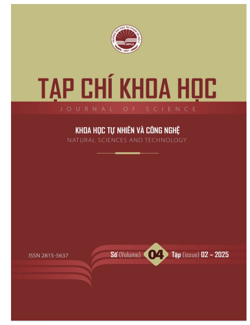

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An identification problem governed by nonlinear fractional mobile-immobile equation, Part I: Solvability

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Abstract

In this study, we address the inverse source problem of identifying a space-dependent parameter in nonlinear fractional mobile-immobile (FrM-IM) equations. The inverse problem is resolved using supplementary measurements taken at the final time, which are permitted to depend implicitly on the system's state. This work is presented in two parts. In Part I, we first establish regularity estimates for resolvent operators associated with the linear FrM-IM equation under Dirichlet boundary conditions. Due to these estimates, we employ fixed-point arguments and local analysis on Hilbert scales to rigorously prove the existence and uniqueness of solutions to the nonlinear inverse problem. In Part II (to be addressed separately), under sufficient regularity assumptions on the final datum and the governing nonlinearities, we demonstrate that the solution derived in Part I is, in fact, a strong solution. Our analysis advances the theoretical framework for FrM-IM equations by unifying resolvent operator theory with nonlinear fixed-point methods, thereby providing a foundation for addressing inverse problems in nonlocal transport phenomena.

Keywords: Mobile–immobile equation, nonlocal PDE, parameter identification, regularity in time, strong solution

1. Introduction

During the last three decades, inverse/direct problems governed by partial differential equations involving fractional-order derivative operators (FrPDEs) have been extensively investigated and published in many research papers or standard monographs; see, e.g., [1], [2]. It is important to note that these studies stem primarily from the effectiveness of FrPDEs in modeling evolutionary phenomena across physics, chemistry, bioengineering, and other fields, where material memory effects play a crucial

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role [3]–[5]. Some typical models that can be mentioned here include the anomalous diffusion equations [6]–[10], the Rayleigh-Stokes equations [11], [12] or the sub-diffusion equations [13], [14], among others.

One of the extensively studied FrPDEs is the fractional mobile-immobile equation, which is given as follows:

$$\nu_1 \partial_t u + \nu_2 \partial_t^\alpha u - \Delta u = S(t,x,u), \text{ in } \Omega, t \in (0,T], \tag{1}$$

where $\Omega \subset \mathbb{R}^d, d \geq 1$ be a bounded domain with smooth boundary $\partial\Omega$, $\nu_1, \nu_2 > 0$, Δ denotes the Laplacian, the notation ∂_t^α represents the Caputo derivative of order α, restricted such that $0 < \alpha < 1$, taken concerning the time variable t and defined as follows:

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(x,s) ds, x \in \Omega, t > 0,$$

and S is an external force.

Let us begin by briefly reviewing some basic facts related to Eq. (1). It is noted that this model was first proposed in the seminal work by R. Schumer and his coauthors [15]. As discussed in the reference [15], Equation (1) is utilized to represent the anomalous diffusion of solute within porous media. Nowadays, there is a considerable number of publications dealing with the existence of numerical solutions for linear as well as nonlinear FrM-IM equations, see e.g., [16]–[18]. Regarding the qualitative properties of solutions to (1), we highlight the recent studies presented in [19], [20]. In [20], the authors have successfully established results on the existence, regularity in time, and stability in the Lyapunov sense of solutions to the Cauchy problem. In addition, the existence of decay solutions to (1) subject to impulsive effects has been obtained in [19].

Nevertheless, it is widely recognized that for numerous practical applications, the forcing function S appearing on the right-hand side of equation (1) is incompletely specified or entirely unknown based on the measured data. In such scenarios, identifying the missing information becomes necessary. Thus the problem of identifying the unknown parameters from suitable observations is of great interest to many researchers. See, e.g., [12], [21]–[24] and the references given therein. It is worth noting that, the identification problem naturally appears in various physical phenomena, such as melting and freezing processes [24], [25]; groundwater hydrology, structural mechanics [22], [26], etc., and is closely related to control theory and optimal design, as remarked in [24], [27]. In this work, we address the situation that the source function S is of the form

$$S(t,x,u) = z(x)h(t) + f(t,u),$$

here, f is a mass/energy-dependent nonlinear perturbation, h is the source strength, and the unknown parameter z characterizes the source's spatial distribution—information not obtainable through direct measurements. More precisely, our goal is to study the problem (**IP**): *Seek the unknown term z, along with the state u obeying the system*

$$\nu_1 \partial_t u + \nu_2 \partial_t^\alpha u - \Delta u = z(x)h(t) + f(t,u), \text{ in } \Omega, t \in (0,T], \tag{2}$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0, \tag{3}$$

$$u(\cdot,0) = \xi \text{ in } \Omega, \tag{4}$$

and the terminal measurement

$$u(x,T) = \varphi(u)(x), x \in \Omega, \tag{5}$$

where $\xi \in L^2(\Omega)$, and h, f, φ are given functions, to be specified in the next section.

As mentioned above, although there are many notable contributions on mobile-immobile equations, the identification problem like (**IP**) has not been studied in the literature, to the best of our knowledge. This serves as the primary motivation for the present study.

Regarding our problem, we are mainly interested in finding sufficient conditions on the nonlinearities f, φ and the real-valued function h to derive conclusions about the existence, Lipschitz-type stability of the solution mapping, and regularity. To address the solvability of the problem (**IP**), we begin by finding an implicit representation of solutions, followed by reformulating the solvability question as a fixed-point problem using the nonlinear nonlocal integral operator $\mathcal{N}$; see Eq. (34) below for the definition of $\mathcal{N}$. It should be noticed here that in our problem, the constraint (5) can be seen as an extension of the final overdetermination, which is usually used in previous works [2], [22], [28] and, in particular, this setting allows the observations to be implicitly dependent on the state. This fact, together with the nonlinearity of f, and the lack of the semigroup property of resolvent operators, leads to some substantial difficulties in our analysis. The main trouble is that z is nonlinearly dependent on u. To overcome these difficulties, we make use of the smoothness in both time and space of resolvent operators, which is established in [20], together with the assumptions that f, φ are locally Lipschitz functions, to create suitable estimates for $z, \mathcal{N}$. With these facts in hand, the existence of solutions for the problem (**IP**) is proved by employing fixed-point point arguments. Additionally, if the functions $h, f,$ and φ possess greater regularity, it is proven that the solution to (**IP**) is strong. Conversely, when φ is independent of u, we further derive a Lipschitz-type stability result for the solution map.

The structure of this paper is organized as follows. In Section 2, we establish foundational results to derive a solution representation for the problem (**IP**) and introduce a sufficient condition on the given data to guarantee existence and uniqueness. Section 3 is dedicated to proving our main results. Here, we employ key techniques such as a priori estimates for local solutions on Hilbert scales and regularity estimates in the time-space framework of two resolvent families. Additionally, we provide remarks on the case where the nonlinearities f and φ satisfy a global Lipschitz condition.

2. Preliminaries

This section commences by revisiting some fundamental concepts concerning the linear direct problems associated with the FrM-IM. These concepts are largely based on the material presented in [20, Sect. 2], with [19] also providing relevant context.

Throughout this work, we use the notations $(\cdot,\cdot), \|\cdot\|$ to designate the inner product and the standard norm respectively, on $L^2(\Omega)$. The sup norms in $C([0,T];L^2(\Omega))$, $C[0,T]$ will be denoted by $\|\cdot\|_\infty$. Besides, we will write $u(t)$ instead of $u(\cdot,t)$ and regard $u(t)$ taking values in $L^2(\Omega)$ for all $t \in [0,T]$.

We will next examine the initial-boundary value problem:

$$\nu_1 \partial_t u + \nu_2 \partial_t^\alpha u - \Delta u = F \text{ in } \Omega, t > 0, \tag{6}$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0, \tag{7}$$

$$u(\cdot,0)=\xi \text{ in } \Omega, \tag{8}$$

where $F\in L^1_{\mathrm{loc}}\left(\mathbb{R}^+;L^2(\Omega)\right)$. Let us now introduce the sequence $\{e_n\}_{n=1}^\infty$, which constitutes an orthonormal basis for the Hilbert space $L^2(\Omega)$. Each element e_n is an eigenfunction of the negative Laplacian operator $(-\Delta)$ within the domain Ω, satisfying homogeneous Dirichlet boundary conditions. This relationship is defined by the eigenvalue problem $-\Delta e_n=\lambda_n e_n$ in Ω, with the condition $e_n=0$ on the boundary $\partial\Omega$. The corresponding eigenvalues $\{\lambda_n\}_{n=1}^\infty$ form a sequence of positive numbers that is strictly increasing and diverges to infinity as $n\to\infty$ (i.e., $\lambda_n>0$ and $\lambda_n\to\infty$). Further details and justification can be found, for example, in [29, Sect. 6.5, p. 354].

For $\gamma\in\mathbb{R}$, the fractional power operator $(-\Delta)^\gamma$ is defined as follows

$$(-\Delta)^\gamma v=\sum_{n=1}^\infty \lambda_n^\gamma (v,e_n)e_n,\ \ D((-\Delta)^\gamma)=\{v\in L^2(\Omega):\sum_{n=1}^\infty \lambda_n^{2\gamma}(v,e_n)^2<\infty\}.$$

Denote $\mathbb{V}_\gamma=D((-\Delta)^\gamma)$. It should be noted that $\mathbb{V}_\gamma$ is a Hilbert space equipped with the norm

$$\|z\|_{\mathbb{V}_\gamma}=\left(\sum_{n=1}^\infty \lambda_n^{2\gamma}\left(z,e_n\right)^2\right)^{\frac{1}{2}},\ z\in D((-\Delta)^\gamma).$$

Furthermore, for each $\gamma>0$, we can identify $\mathbb{V}_{-\gamma}=D((-\Delta)^{-\gamma})$ with $\mathbb{V}_\gamma^*$, the dual space of $\mathbb{V}_\gamma$.

Assume that $u(t)=\sum_{n=1}^\infty u_n(t)e_n, F(t)=\sum_{n=1}^\infty F_n(t)e_n$. Substituting into (6)–(8), we find that

$$\nu_1 u_n'(t)+\nu_2 g_{1-\alpha}*u_n'(t)+\lambda_n u_n(t)=F_n(t), t>0, \tag{9}$$

$$u_n(0)=\xi_n:=(\xi,e_n). \tag{10}$$

In this context, '$*$' represents the temporal Laplace convolution, calculated as $(a*v)(t)=\int_0^t a(t-s)v(s)ds,$ and $g_{1-\alpha}(t)=t^{-\alpha}/\Gamma(1-\alpha), t>0$.

To find u_n that satisfies Eqs. (9)–(10), our approach now involves considering the following scalar integral equations

$$s(t)+\lambda(\ell*s)(t)=1, t\ge 0, \tag{11}$$

$$r(t)+\lambda(\ell*r)(t)=\ell(t), t\ge 0, \tag{12}$$

where $\lambda>0$ and ℓ is the unique solution to the following integral equation

$$\nu_1\ell+\nu_2 g_{1-\alpha}*\ell=1 \text{ on } [0,\infty). \tag{13}$$

It is well known (see, e.g., [30, Theorem 2.3.1]), that Eqs. (11) and (12) are uniquely solved. In particular, see [20, Sect. 2], the solution of Eq. (13) is given by

$$\ell(t)=\nu_1^{-1}E_{1-\alpha}(-\nu_1^{-1}\nu_2 t^{1-\alpha}), \tag{14}$$

where $E_{1-\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma((1-\alpha)n+1)}, z\in\mathbb{C}$, is the Mittag–Leffler function. Throughout this work, we denote $s_\alpha(\cdot,\lambda)$ and $r_\alpha(\cdot,\lambda)$ being the solutions of (11) and (12), respectively. Recall that the kernel function ℓ is completely positive iff $s_\alpha(\cdot,\lambda), r_\alpha(\cdot,\lambda)$ are nonnegative for every $\lambda>0$, see [31]. In [32, Proposition 3.23, p. 47], it is shown that ℓ is completely positive. Furthermore, by applying the same line of reasoning found in [20, Propositions 2.1 and 2.2], we arrive at the subsequent results.

Proposition 2.1. *Suppose ℓ, $s_\alpha(\cdot,\lambda)$, and $r_\alpha(\cdot,\lambda)$ represent the respective solutions to equations (13), (11), and (12). Consequently, the following properties hold:*

(i) The function $\ell(t)$ is bounded as follows:

$$\frac{1}{\nu_1+\nu_2\Gamma(\alpha)t^{1-\alpha}}\le \ell(t)\le \frac{1}{\nu_1+\nu_2\Gamma(2-\alpha)^{-1}t^{1-\alpha}}, \text{ for all } t\ge 0.$$

(ii) $\ell(\cdot)$ is differentiable on the interval $(0,\infty)$, and its derivative satisfies

$$0\le -\ell'(t)\le \nu_1^{-2}\nu_2 t^{-\alpha} \text{ for almost every } t>0.$$

(iii) For any given $\lambda>0$, the function $s_\alpha(\cdot,\lambda)$ is non-negative and nonincreasing. Additionally, the inequality holds:

$$s_\alpha(t,\lambda)\left[1+\lambda\int_0^t \ell(\tau)d\tau\right]\le 1,\ \forall\ t\ge 0. \tag{15}$$

(iv) For each fixed $t>0$, the functions $\lambda\mapsto s_\alpha(t,\lambda)$ and $\lambda\mapsto r_\alpha(t,\lambda)$ are nonincreasing with respect to λ.

(v) The function $r_\alpha(\cdot,\lambda)$ is non-negative, and the subsequent two equalities are valid:

$$s_\alpha(t,\lambda)=1-\lambda\int_0^t r_\alpha(\tau,\lambda)d\tau=\nu_1 r_\alpha(t,\lambda)+\nu_2(g_\alpha * r_\alpha(\cdot,\lambda))(t), t\ge 0. \tag{16}$$

Furthermore, for every $\lambda>0$, the estimates below are satisfied:

$$\lambda r_\alpha(t,\lambda)\le \frac{1}{t},\ \forall t>0 \text{ and } r_\alpha(t,\lambda)\le \ell(t),\ \forall t\ge 0. \tag{17}$$

Remark 2.1. (i) Given the representation (16) and the inequality (17), for each $\lambda>0$, we have

$$0\le -s_\alpha'(t,\lambda)=\lambda r_\alpha(t,\lambda)\le\frac{1}{t}, \text{ for all } t>0, \text{ and } \lambda\int_0^t r_\alpha(s,\lambda)ds\le 1, \forall t\ge 0.$$

(ii) Let $v(t)=s_\alpha(t,\lambda)v_0+\left(r_\alpha(\cdot,\lambda)*\omega\right)(t)$, here $\omega\in L^1_{loc}(\mathbb{R}^+)$. Then, following the same line of reasoning as in [20, Proposition 2.3], it can be shown that v solves the problem

$$\nu_1 v'(t)+\nu_2\left(g_{1-\alpha}*v'\right)(t)+\lambda v(t)=\omega(t), v(0)=v_0. \tag{18}$$

It follows from Remark 2.1(ii) and Eqs. (9)–(10) that $u_n(t)=s_\alpha(t,\lambda_n)\xi_n+r_\alpha(\cdot,\lambda_n)*F_n(t)$. We then obtain

$$u(t)=\mathcal{S}_\alpha(t)\xi+\int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau, \tag{19}$$

where

$$\mathcal{S}_\alpha(t)v=\sum_{n=1}^{\infty}s_\alpha(t,\lambda_n)(v,e_n)e_n, v\in L^2(\Omega), \tag{20}$$

$$\mathcal{R}_\alpha(t)v=\sum_{n=1}^{\infty}r_\alpha(t,\lambda_n)(v,e_n)e_n, v\in L^2(\Omega). \tag{21}$$

As evident from formulas (20) and (21), $\mathcal{S}_\alpha(t)$ and $\mathcal{R}_\alpha(t)$ are linear operators on $L^2(\Omega)$. The following lemma summarizes some of their fundamental properties.

Lemma 2.1 *(See [19, Lemma 2.4]). Consider the operator families* $\{\mathcal{S}_\alpha(t)\}_{t\ge 0}$ *and* $\{\mathcal{R}_\alpha(t)\}_{t\ge 0}$ *as given in (20) and (21). The following properties hold:*

(i) For any $v\in L^2(\Omega)$ *and* $T>0$, *the function* $t\mapsto \mathcal{S}_\alpha(t)v$ *is continuous on $[0,T]$ with values in* $L^2(\Omega)$, *whereas* $t\mapsto \Delta\mathcal{S}_\alpha(t)v$ *is continuous on* $(0,T]$. *The norms obey:*

$$\|\mathcal{S}_\alpha(t)v\|\le s_\alpha(t,\lambda_1)\|v\|, t\in[0,T], \tag{22}$$

$$\|\mathcal{S}_\alpha(t)v\|_{\mathbb{V}_1}\le\frac{\|v\|}{(1*\ell)(t)}, t\in(0,T]. \tag{23}$$

The operator $\mathcal{S}_\alpha(\cdot)$ *is differentiable for* $t>0$, *satisfying:*

$$\|\mathcal{S}_\alpha'(t)v\|\le\frac{\|v\|}{t},\forall v\in L^2(\Omega),\forall t>0. \tag{24}$$

(ii) If $v\in L^2(\Omega), T>0,$ *and* $g\in C([0,T];L^2(\Omega)),$ *then* $\mathcal{R}_\alpha(\cdot)v$ *is continuous on* $[0,T]$, *and the convolution* $\mathcal{R}_\alpha * g$ *maps into* $C([0,T];\mathbb{V}_{1/2})$. *The following bounds apply:*

$$\|\mathcal{R}_\alpha(t)v\|\le r_\alpha(t,\lambda_1)\|v\|, t\in[0,T], \tag{25}$$

$$\|(\mathcal{R}_\alpha * g)(t)\|\le\int_0^t r_\alpha(t-\tau,\lambda_1)\|g(\tau)\|d\tau, t\in[0,T], \tag{26}$$

$$\|(\mathcal{R}_\alpha * g)(t)\|_{\mathbb{V}_{1/2}}\le\left(\int_0^t r_\alpha(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}}, t\in[0,T]. \tag{27}$$

Moreover, $\mathcal{R}_\alpha(\cdot)$ *is differentiable for* $t>0$, *with the derivative estimate:*

$$\|\mathcal{R}_\alpha'(t)v\|\le(\nu_1^{-1}t^{-1}+\nu_1^{-2}\nu_2 t^{-\alpha})\|v\|,\forall v\in L^2(\Omega),\forall t>0. \tag{28}$$

In order to solve the inverse problem (**IP**), we require the following conditions:

(H1) The real function h is continuous on $[0,T]$ and $m:=\inf_{[0,T]}h(t)>0$.

(H2) The nonlinearity $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ satisfies $f(\cdot,0)=0$, and is locally Lipschitz with respect to the second variable, that is, for each $r>0$, there exists $L_f(r)>0$ such that

$$\|f(t,v_1)-f(t,v_2)\|\le L_f(r)\|v_1-v_2\|,\forall t\in[0,T],\|v_1\|,\|v_2\|\le r.$$

(H3) The function $\varphi: C([0,T]; L^2(\Omega)) \to \mathbb{V}_{1/2}$ satisfies $\varphi(0)=0$ and is locally Lipschitz continuous, that is, for each $r>0$, there exists $L_\varphi(r)>0$ such that

$$\| \varphi(\omega_1)-\varphi(\omega_2) \|_{\mathbb{V}_{1/2}} \le L_\varphi(r) \| \omega_1-\omega_2 \|_\infty, \forall \| \omega_1 \|_\infty, \| \omega_2 \|_\infty \le r.$$

Subsequently, we construct a representation of solutions for the inverse source problem. Assuming (u,z) is a solution to the problem (IP), we set $z_n=(z,e_n)_{\mathbb{V}_{-1/2},\mathbb{V}_{1/2}}$, $f_n(t)=\big(f(t,u(t)),e_n\big)$, and $\varphi_n=(\varphi(u),e_n)$. Then

$$u(t)=\sum_{n=1}^{\infty}\Big[s_\alpha(t,\lambda_n)\xi_n+\int_0^t r_\alpha(t-\tau,\lambda_n)\big(z_n h(\tau)+f_n(\tau)\big)d\tau\Big].$$

By employing the terminal data (4) and matching the coefficients, we obtain:

$$\varphi_n = s_\alpha(T,\lambda_n)\xi_n+\int_0^T r_\alpha(T-\tau,\lambda_n)\big(z_n h(\tau)+f_n(\tau)\big)d\tau.$$

Hence

$$z_n=\Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-1}\Big[\varphi_n-s_\alpha(T,\lambda_n)\xi_n-\int_0^T r_\alpha(T-\tau,\lambda_n)f_n(\tau)d\tau\Big].$$

Let $\mathbb{Q}: L^2(\Omega)\to L^2(\Omega)$ be the operator defined by

$$\mathbb{Q}v=\sum_{n=1}^{\infty}\Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-1} v_n e_n, v\in L^2(\Omega),$$

$$D(\mathbb{Q})=\Big\{v\in L^2(\Omega): \sum_{n=1}^{\infty}\Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-2} v_n^2<\infty\Big\}.$$

The arguments above lead to

$$u(t)=\mathcal{S}_\alpha(t)\xi+\int_0^t \mathcal{R}_\alpha(t-\tau)\big(zh(\tau)+f(\tau,u(\tau))\big)d\tau, t\in[0,T], \tag{29}$$

$$z=\mathbb{Q}\Big[\varphi(u)-\mathcal{S}_\alpha(T)\xi-\int_0^T \mathcal{R}_\alpha(T-\tau)f(\tau,u(\tau))d\tau\Big]. \tag{30}$$

Employing the formulation of $\mathbb{Q}$ and Proposition 2.1, we obtain the following lemma which will be useful for the proof of the main results.

Lemma 2.2. Suppose (H1) holds and let m_T be defined as $m_T=m\big(1-s_\alpha(T,\lambda_1)\big)$. Then the subsequent assertions are valid:

(i) $D(\mathbb{Q})=D(-\Delta)$;

(ii) For any $v\in\mathbb{V}_1$, it follows that $\mathbb{Q}v\in L^2(\Omega)$ and the inequality $\|\mathbb{Q}v\|\le m_T^{-1}\lambda_1^{-1}\|v\|_{\mathbb{V}_1}$ is satisfied;

(iii) If $v\in\mathbb{V}_{1/2}$, then $\mathbb{Q}v\in\mathbb{V}_{-1/2}$ and the estimate $\|\mathbb{Q}v\|_{\mathbb{V}_{-1/2}}\le m_T^{-1}\|v\|_{\mathbb{V}_{1/2}}$ holds;

(iv) Given $\phi\in C([0,T];L^2(\Omega))$, we have $\mathbb{Q}(\mathcal{R}_\alpha*\phi)(T)\in\mathbb{V}_{-1/2}$ and

$$\|\mathbb{Q}(\mathcal{R}_\alpha*\phi)(T)\|_{\mathbb{V}_{-1/2}}\le m_T^{-1}\lambda_1^{-1/2}\|\phi\|_\infty .$$

Proof. (i) By Proposition 2.1, one has

$$\begin{aligned}\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau &\geq m\int_0^T r_\alpha(\tau,\lambda_n)d\tau\\ &= m\lambda_n^{-1}\big(1-s_\alpha(T,\lambda_n)\big)\\ &\geq m\big(1-s_\alpha(T,\lambda_1)\big)\lambda_n^{-1}\\ &= m_T\lambda_n^{-1},\end{aligned}$$

and

$$\begin{aligned}\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau &\leq \| h\|_\infty \int_0^T r_\alpha(\tau,\lambda_n)d\tau\\ &= \| h\|_\infty \lambda_n^{-1}\big(1-s_\alpha(T,\lambda_n)\big)\\ &\leq \| h\|_\infty \lambda_n^{-1}.\end{aligned}$$

Based on the preceding estimates, it follows that for any $v\in D(\mathbb{Q})$, we have:

$$\sum_{n=1}^{\infty}\Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-2}v_n^2 \geq \sum_{n=1}^{\infty}\| h\|_\infty^{-2}\lambda_n^2 v_n^2.$$

This implies v belongs to $D(-\Delta)$. Conversely, whenever v is in $D(-\Delta)$, the following holds:

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-2}v_n^2 &\leq \sum_{n=1}^{\infty} m_T^{-2}\lambda_n^2 v_n^2\\ &= m_T^{-2}\,\|\Delta v\|^2.\end{aligned}$$

The later inequality implies that $v\in D(\mathbb{Q})$.

(ii) Suppose $v\in \mathbb{V}_1 = D(-\Delta)$. By part (i), it follows that $v\in D(\mathbb{Q})$ and furthermore $\|\mathbb{Q}v\|^2 \leq m_T^{-2}\,\|\Delta v\|^2$. Subsequently, applying the fact $\| v\| \leq \lambda_1^{-\beta}\,\| v\|_{\mathbb{V}_\beta}, \beta>0, v\in L^2(\Omega)$ to the preceding inequality results in $\|\mathbb{Q}v\| \leq m_T^{-1}\lambda_1^{-1}\,\| v\|_{\mathbb{V}_1}$.

(iii) If $v\in \mathbb{V}_{1/2}$, then we have

$$\begin{aligned}\|(-\Delta)^{-1/2}\mathbb{Q}v\|^2 &= \sum_{n=1}^{\infty}\Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-2}\lambda_n^{-1}v_n^2\\ &\leq \sum_{n=1}^{\infty} m_T^{-2}\lambda_n v_n^2.\end{aligned}$$

Thus $\|\mathbb{Q}v\|_{\mathbb{V}_{1/2}} \leq m_T^{-1}\,\| v\|_{\mathbb{V}_{1/2}}$.

(iv) Consider $\phi\in C([0,T];L^2(\Omega))$ and define $\phi_n(t)=(\phi(t),e_n)$ for $t\geq 0$. Applying the Hölder inequality yields:

$$\begin{aligned}\| \mathbb{Q}(\mathcal{R}_\alpha * \phi)(T) \|^2_{\mathbb{V}_{-1/2}} &= \sum_{n=1}^{\infty} \Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-2} \lambda_n^{-1} \Big(\int_0^T r_\alpha(T-\tau,\lambda_n)\phi_n(\tau)d\tau\Big)^2 \\ &\le \sum_{n=1}^{\infty} m_T^{-2}\lambda_n \int_0^T r_\alpha(T-\tau,\lambda_n)d\tau \int_0^T r_\alpha(T-\tau,\lambda_n)\,|\,\phi_n(\tau)|^2\, d\tau \\ &= \sum_{n=1}^{\infty} m_T^{-2}\Big(1-s_\alpha(T,\lambda_n)\Big)\int_0^T r_\alpha(T-\tau,\lambda_n)\,|\,\phi_n(\tau)|^2\, d\tau \\ &\le m_T^{-2}\int_0^T r_\alpha(T-\tau,\lambda_1)\,\|\,\phi(\tau)\|^2\, d\tau \le m_T^{-2}\lambda_1^{-1}\,\|\,\phi\,\|_\infty^2,\end{aligned}$$

which verifies condition (iv) and consequently concludes the proof of Lemma 2.2. □

Motivated by (29)–(30), we now define what constitutes a solution to the inverse problem (**IP**).

Definition 2.1. For a given $\xi \in L^2(\Omega)$, the pair $(u,z) \in C([0,T];L^2(\Omega)) \times \mathbb{V}_{-1/2}$ is said to be a solution to the problem (**IP**) if

$$u(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)\Big(zh(\tau) + f(\tau,u(\tau))\Big)d\tau, t \in [0,T], \tag{31}$$

$$z = \mathbb{Q}\Big[\varphi(u) - \mathcal{S}_\alpha(T)\xi - \int_0^T \mathcal{R}_\alpha(T-\tau)f(\tau,u(\tau))d\tau\Big]. \tag{32}$$

3. Results on the solvability of the problem (IP)

We are now in a position to state the first main result on the existence for the identification problem (**IP**).

Theorem 3.1. *Suppose that hypotheses (H1)–(H3) are satisfied. Then there exists $\rho > 0$ such that the inverse source problem (**IP**) admits a unique solution, provided that*

$$\|\,h\,\|_\infty\, m_T^{-1}\lambda_1^{1/2} \limsup_{r\to 0} L_\varphi(r) + (\|\,h\,\|_\infty\, m_T^{-1} + 1)\limsup_{r\to 0} L_f(r) < \lambda_1, \tag{33}$$

and $\|\,\xi\,\| \le \rho$.

Proof of Theorem 3.1. Put

$$\limsup_{r\to 0} L_\varphi(r) = \alpha_\varphi, \limsup_{r\to 0} L_f(r) = \alpha_f.$$

By our assumption (33), one can select $\epsilon > 0$ such that

$$\|\,h\,\|_\infty\, m_T^{-1}\lambda_1^{1/2}(\alpha_\varphi + \epsilon) + \Big(\|\,h\,\|_\infty\, m_T^{-1} + 1\Big)(\alpha_f + \epsilon) < \lambda_1.$$

Moreover, the definition of $\limsup$ ensures that we can find an $R > 0$ for which

$$L_\varphi(r) \le \alpha_\varphi + \epsilon, L_f(r) \le \alpha_f + \epsilon, \forall r \in [0,R].$$

To complete the proof, we now need to demonstrate that the operator $\mathcal{N}$, which is defined by

$$\mathcal{N}(u)(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)[zh(\tau) + f(\tau,u(\tau))]d\tau, t \in [0,T], \tag{34}$$

with

$$z = z(u) = \mathbb{Q}\Big[\varphi(u) - \mathcal{S}_\alpha(T)\xi - \int_0^T \mathcal{R}_\alpha(T-\tau)f(\tau,u(\tau))d\tau\Big], \tag{35}$$

possesses a unique fixed point within B_R, where B_R represents the closed ball in $C([0,T];L^2(\Omega))$ centered at the origin with radius R.

Our proof is quite long, we divide it into several assertions for the sake of clarity.

Step 1. Estimate of z. Considering $u \in B_R$, we obtain

$$\| z \|_{\mathbb{V}_{-1/2}} \leq \| \mathbb{Q}\varphi(u) \|_{\mathbb{V}_{-1/2}} + \| \mathbb{Q}\mathcal{S}_\alpha(T)\xi \|_{\mathbb{V}_{-1/2}} + \| \mathbb{Q}\mathcal{R}_\alpha * f(\cdot,u(\cdot))(T) \|_{\mathbb{V}_{-1/2}} := I_1 + I_2 + I_3.$$

Owing to Lemma 2.2(iii), we get that

$$I_1 \leq m_T^{-1} \| \varphi(u) \|_{\mathbb{V}_{1/2}} \leq m_T^{-1} L_\varphi(R)R \leq m_T^{-1}(\alpha_\varphi + \epsilon)R. \tag{36}$$

As for I_2, we have

$$\| \mathbb{Q}\mathcal{S}_\alpha(T)\xi \|^2_{\mathbb{V}_{-1/2}} = \sum_{n=1}^{\infty} \Big(\int_0^T r_\alpha(T-\tau,\lambda_n)h(\tau)d\tau\Big)^{-2} \lambda_n^{-1} s_\alpha(T,\lambda_n)^2 \xi_n^2$$
$$\leq m_T^{-2} \sum_{n=1}^{\infty} \lambda_n s(T,\lambda_n)^2 \xi_n^2 = m_T^{-2} \| (-\Delta)^{1/2} \mathcal{S}_\alpha(T)\xi \|^2 .$$

Thus, using Lemma 2.1(i) gives

$$I_2 \leq m_T^{-1} \| (-\Delta)^{1/2} \mathcal{S}_\alpha(T)\xi \| \leq m_T^{-1}\lambda_1^{-1/2} \| \Delta \mathcal{S}_\alpha(T)\xi \| \leq m_T^{-1}\lambda_1^{-1/2}(1*\ell)(T)^{-1} \| \xi \|. \tag{37}$$

Concerning I_3, on one hand

$$\| f(t,u(t)) \| \leq L_f(R)R \leq (\alpha_f + \epsilon)R, \forall t \in [0,T], \tag{38}$$

and on the other

$$I_3 \leq m_T^{-1}\lambda_1^{-1/2} \| f(\cdot,u(\cdot)) \|_\infty \leq m_T^{-1}\lambda_1^{-1/2}(\alpha_f + \epsilon)R,$$

thanks to Lemma 2.2(iv). Combining all estimates (36), (37), and (38), we conclude that

$$\| z \|_{\mathbb{V}_{-1/2}} \leq m_T^{-1}\big[(\alpha_\varphi + \epsilon)R + \lambda_1^{-1/2}(\alpha_f + \epsilon)R + \lambda_1^{-1/2}(1*\ell)(T)^{-1} \| \xi \|\big]. \tag{39}$$

Step 2. Estimate of $\mathcal{R}_\alpha * (zh)$. Note that

$$\| \mathcal{R}_\alpha * (zh)(t) \|^2 = \sum_{n=1}^{\infty} \Big(\int_0^t r_\alpha(t-\tau,\lambda_n) z_n h(\tau)d\tau\Big)^2$$
$$\leq \sum_{n=1}^{\infty} \Big(\int_0^t r_\alpha(t-\tau,\lambda_n)d\tau\Big)^2 z_n^2 \| h \|^2_\infty \leq \sum_{n=1}^{\infty} \lambda_n^{-2} z_n^2 \| h \|^2_\infty .$$

Therefore

$$\| \mathcal{R}_\alpha * (zh)(t) \| \leq \| h \|_\infty \lambda_1^{-1/2} \| z \|_{\mathbb{V}_{-1/2}} \leq \| h \|_\infty \lambda_1^{-1/2} \| z \|_{\mathbb{V}_{-1/2}}, \forall t \in [0,T]. \tag{40}$$

Step 3. Estimate of $\mathcal{N}(u)$. Using Lemma 2.1 and the estimates provided in (39)–(40), we obtain the following:

$$\begin{aligned}\|\mathcal{N}(u)(t)\| &\le s_\alpha(t,\lambda_1)\|\xi\| + \|\mathcal{R}_\alpha * (zh)(t)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)L_f(R)\|u(\tau)\|d\tau \\ &\le \|\xi\| + \|\mathcal{R}_\alpha * (zh)(t)\| + (\alpha_f+\epsilon)R\int_0^T r_\alpha(\tau,\lambda_1)d\tau \\ &\le \|\xi\| + \|\mathcal{R}_\alpha * (zh)(t)\| + (\alpha_f+\epsilon)\lambda_1^{-1}R \\ &\le \|\xi\|\left(1+\|h\|_\infty \lambda_1^{-1}m_T^{-1}(1*\ell)(T)^{-1}\right) \\ &\quad + \left[\|h\|_\infty m_T^{-1}\lambda_1^{-1/2}(\alpha_\varphi+\epsilon) + \left(\|h\|_\infty m_T^{-1}+1\right)\lambda_1^{-1}(\alpha_f+\epsilon)\right]R.\end{aligned}$$

From the preceding inequality, we deduce that if $\rho > 0$ is chosen satisfying

$$\rho \le (1+\|h\|_\infty \lambda_1^{-1}m_T^{-1}(1*\ell)(T)^{-1})^{-1}\times[1-\|h\|_\infty m_T^{-1}\lambda_1^{-1/2}(\alpha_\varphi+\epsilon)+(\|h\|_\infty m_T^{-1}+1)\lambda_1^{-1}(\alpha_f+\epsilon)]R,$$

then for any $u \in B_R$ such that $\|\xi\| \le \rho,$ it holds that $\mathcal{N}(u) \in B_R$.

By the discussion above, we now consider the operator $\mathcal{N}$ on B_R. Due to the representation (32), for all $u_1, u_2 \in B_R$, it holds that

$$\begin{aligned}\|z(u_1)-z(u_2)\|_{\mathbb{V}_{-1/2}} &\le \|\mathbb{Q}[\varphi(u_1)-\varphi(u_2)]\|_{\mathbb{V}_{-1/2}} \\ &\quad + \|\mathbb{Q}\mathcal{R}_\alpha * \left(f(\cdot,u_1(\cdot))-f(\cdot,u_2(\cdot))\right)(T)\|_{\mathbb{V}_{-1/2}}.\end{aligned} \tag{41}$$

Using an argument analogous to the one used for I_1, I_3, it can be shown that

$$\|\mathbb{Q}[\varphi(u_1)-\varphi(u_2)]\|_{\mathbb{V}_{-1/2}} \le m_T^{-1}(\alpha_\varphi+\epsilon)\|u_1-u_2\|_\infty, \tag{42}$$

and

$$\|\mathbb{Q}\mathcal{R}_\alpha * \left(f(\cdot,u_1(\cdot))-f(\cdot,u_2(\cdot))\right)(T)\|_{\mathbb{V}_{-1/2}} \le m_T^{-1}\lambda_1^{-1/2}(\alpha_f+\epsilon)\|u_1-u_2\|_\infty, \tag{43}$$

and

$$\begin{aligned}\|\mathcal{R}_\alpha * (z(u_1)-z(u_2))h\|_\infty &\le \|h\|_\infty \lambda_1^{-1/2}\|z(u_1)-z(u_2)\|_{\mathbb{V}_{-1/2}} \\ &\le \|h\|_\infty \lambda_1^{-1/2}\left[m_T^{-1}(\alpha_\varphi+\epsilon)+m_T^{-1}\lambda_1^{-1/2}(\alpha_f+\epsilon)\right]\|u_1-u_2\|_\infty.\end{aligned} \tag{44}$$

Combining (42), (43), (44) together with (41), one can see that

$$\begin{aligned}&\|\mathcal{N}(u_1)-\mathcal{N}(u_2)\|_\infty \\ &\le \|\mathcal{R}_\alpha * (z(u_1)-z(u_2))h\|_\infty + \|\mathcal{R}_\alpha * \left(f(\cdot,u_1(\cdot))-f(\cdot,u_2(\cdot))\right)\|_\infty \\ &\le \|h\|_\infty \lambda_1^{-1/2}\left[m_T^{-1}(\alpha_\varphi+\epsilon)+m_T^{-1}\lambda_1^{-1/2}(\alpha_f+\epsilon)\right]\|u_1-u_2\|_\infty + \lambda_1^{-1}(\alpha_f+\epsilon)\|u_1-u_2\|_\infty \\ &= \left[\|h\|_\infty m_T^{-1}\lambda_1^{-1/2}(\alpha_\varphi+\epsilon)+\left(\|h\|_\infty m_T^{-1}+1\right)\lambda_1^{-1}(\alpha_f+\epsilon)\right]\|u_1-u_2\|_\infty.\end{aligned}$$

This final inequality demonstrates that $\mathcal{N}$ is a contraction mapping on B_R. The proof is thus complete. □

We now provide some remarks concerning the feasibility of condition (33) presented in Theorem 3.1. This condition necessitates constraints on the magnitude (smallness) of the nonlinearities f and φ. In the particular scenario where f and φ are globally Lipschitz functions (i.e., $L_f(r)$ and $L_\varphi(r)$ are positive constants), the arguments employed in the proof of Theorem 2.1 can be directly utilized to establish the existence and uniqueness of a solution for the inverse problem (**IP**). Moreover, under these global Lipschitz assumptions, the requirement for a small initial condition, along with the conditions $f(\cdot,0)=0$ and $\varphi(0)=0$, can be relaxed, as detailed in Corollary 2.1 below.

Corollary 3.1. *Let the hypothesis (H1) hold and assume that the nonlinearity functions* f,φ *satisfy*

(H2)$'$ $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ *is globally Lipschitz continuous, that is, there exists* $L_f>0$ *such that*

$$\| f(t,v_1)-f(t,v_2)\|\le L_f \|v_1-v_2\|, \forall t\in[0,T], v_1,v_2\in L^2(\Omega);$$

(H3)$'$ $\varphi: C([0,T];L^2(\Omega))\to \mathbb{V}_{1/2}$ *is globally Lipschitz continuous, that is, there exists* $L_\varphi>0$ *such that*

$$\|\varphi(w_1)-\varphi(w_2)\|_{\mathbb{V}_{1/2}}\le L_\varphi \|w_1-w_2\|_\infty, \forall w_1,w_2\in C([0,T];L^2(\Omega)).$$

*Then the problem (**IP**) has a unique solution, provided that*

$$\|h\|_\infty\, m_T^{-1}\lambda_1^{1/2}L_\varphi+\big(\|h\|_\infty\, m_T^{-1}+1\big)L_f<\lambda_1.$$

To close this section, let us give an example of the nonlinear function f, φ *satisfying the hypotheses (H2) and (H3). Let*

$$f(t,v)(x)=p(t)q\Big(\int_\Omega |v(x)|^2\,dx\Big)v(x), t\in[0,T], v\in L^2(\Omega), \tag{45}$$

$$\varphi(\omega)(x)=\psi\Big(\int_\Omega |\omega(T,y)|^2\,dy\Big)\varphi_T(x), x\in\Omega, \omega\in C([0,T];L^2(\Omega)), \tag{46}$$

where p is continuous function on $[0,T]$, $q\in C^1(\mathbb{R}^+)$ is a function such that $|q(s)|\le\gamma|s|^\sigma$ for some $\gamma>0,\sigma>0$. Note that, for all $v_1,v_2\in L^2(\Omega)$, $\|v_1\|,\|v_2\|\le r$, $t\in[0,T]$, we have that

$$\begin{aligned}&\|f(t,v_1)-f(t,v_2)\|\\ &\le |p(t)|\big[\big|q(\|v_1\|^2)-q(\|v_2\|^2)\big|\|v_2\|+\big|q(\|v_1\|^2)\big|\|v_1-v_2\|\big]\\ &\le \|p\|_\infty\big[(\|v_1\|+\|v_2\|)\|v_2\|\big|q'\big((1-\kappa)\|v_2\|^2+\kappa\|v_1\|^2\big)\big|+\gamma\|v_1\|^{2\sigma}\big]\|v_1-v_2\|,\end{aligned}$$

where $\kappa\in[0,1]$, thanks to the mean value formula. Therefore

$$\|f(t,v_1)-f(t,v_2)\|\le\|p\|_\infty\big[2r^2\sup_{s\in[0,r^2]}|q'(s)|+\gamma r^{2\sigma}\big]\|v_1-v_2\|,$$

which means that f obeys (H2) with

$$L_f(r)=\|p\|_\infty\big[2r^2\sup_{s\in[0,r^2]}|q'(s)|+\gamma r^{2\sigma}\big].$$

Concerning the function φ given by Eq. (46), we assume that $\varphi_T\in H_0^1(\Omega)$ is the desired datum at $t=T$, but the measured datum is subject to a multiplicative perturbation, namely $\psi\Big(\int_\Omega |\omega(T,y)|^2\,dy\Big)$, which depends on the mass of the system. Then one sees that $\varphi(\omega)\in H_0^1(\Omega)=\mathbb{V}_{1/2}$ for each $\omega\in C([0,T];L^2(\Omega))$. Assuming that $\psi\in C^1(\mathbb{R}^+)$, one can show that $\varphi(\cdot)$ fulfills the hypothesis (H3) by the same reasoning as for f.

4. Conclusion

This study has focused on the inverse source problem concerning the identification of a spatially dependent parameter within the right-hand side term of a nonlinear FrM-IM equation. The unknown parameter and the associated state are determined using supplementary observations acquired at the final time. Notably, these observations exhibit an implicit dependence on the state variable, introducing significant technical complexities into the analysis. By imposing suitable regularity and structural assumptions on the governing nonlinearities and final datum, we establish rigorous existence and uniqueness results for solutions to the inverse problem.

This work advances the theoretical understanding of parameter identification in nonlocal transport models governed by FrM-IM dynamics. The developed framework–combining resolvent operator theory, fixed-point methods, and Hilbert-scale analysis–provides a robust foundation for addressing nonlinear inverse problems with implicit measurement dependencies and opens avenues for future applications in anomalous diffusion and subsurface transport modeling.

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