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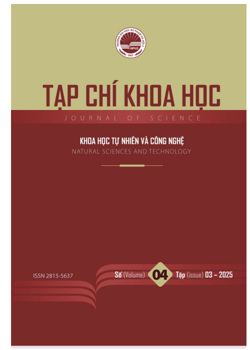

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An identification problem governed by nonlinear fractional mobile-immobile equation, Part II: Stability and Regularity

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Abstract

This paper continues our recent work in [1], in which we investigated the existence and uniqueness of solutions for the inverse problem (**IP**): *Seek the unknown term* z, *along with the state* u *obeying the system*

$$\nu_1\partial_t u + \nu_2\partial_t^\alpha u - \Delta u = z(x)h(t) + f(t,u), \text{ in } \Omega, t \in (0,T], \tag{1}$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0, \tag{2}$$

$$u(\cdot,0) = \xi \text{ in } \Omega, \tag{3}$$

and the terminal measurement

$$u(x,T) = \varphi(u)(x), x \in \Omega. \tag{4}$$

In this setting, $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ denotes a bounded domain whose boundary $\partial\Omega$ is smooth. Motivated by considerations arising from numerical analysis, the primary aim of this work is to establish a set of sufficient conditions on the functions h, f, and φ that guarantee both the continuous dependence of solutions on the data and the regularity in time of the solution pair (u,z) for the inverse problem (**IP**).

Keywords: Mobile–immobile equation, stability, parameter identification, strong solution

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1. Introduction and main results

Before presenting our results, we recall, for the reader's convenience, some notations, conventions, and facts related to the problem (**IP**), as introduced in [1].

Let $L^2(\Omega)$ denote the space of Lebesgue square-integrable measurable functions on Ω. The inner product and standard norm on $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. For $\gamma \geq 0$, the space $\mathbb{V}_\gamma$ represents the domain of the fractional power of the Laplacian with homogeneous Dirichlet boundary conditions, defined as

$$\mathbb{V}_\gamma = D((-\Delta)^\gamma) = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} (v, e_n)^2 < \infty \right\},$$

where $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of the Laplacian $-\Delta$ subject to homogeneous Dirichlet boundary conditions, satisfying

$$-\Delta e_n = \lambda_n e_n, \quad e_n|_{\partial\Omega} = 0,$$

with $\{\lambda_n\}_{n=1}^\infty$ forming an increasing sequence such that $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$. For a normed linear space X, we denote by $C([0,T];X)$ the Banach space of all continuous functions from $[0,T]$ to X. The supremum norm on $C([0,T];L^2(\Omega))$ and $C([0,T])$ is denoted by $\|\cdot\|_\infty$.

As established in [1, Theorem 3.1], if the following hypotheses hold:

(H1) The real function h is continuous on $[0,T]$ and satisfies $m := \inf_{[0,T]} h(t) > 0$;

(H2) The nonlinearity $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies $f(\cdot,0) = 0$, and is locally Lipschitz with respect to the second variable, that is, for each $r > 0$, there exists $L_f(r) > 0$ such that

$$\| f(t,v_1) - f(t,v_2) \| \leq L_f(r) \| v_1 - v_2 \|, \forall t \in [0,T], \| v_1 \|, \| v_2 \| \leq r;$$

(H3) The function $\varphi : C([0,T];L^2(\Omega)) \to \mathbb{V}_{1/2}$ satisfies $\varphi(0) = 0$ and is locally Lipschitz continuous, that is, for each $r > 0$, there exists $L_\varphi(r) > 0$ such that

$$\| \varphi(\omega_1) - \varphi(\omega_2) \|_{\mathbb{V}_{1/2}} \leq L_\varphi(r) \| \omega_1 - \omega_2 \|_\infty, \forall \| \omega_1 \|_\infty, \| \omega_2 \|_\infty \leq r;$$

(H4) The following inequality holds:

$$\| h \|_\infty m_T^{-1} \lambda_1^{1/2} \limsup_{r \to 0} L_\varphi(r) + (\| h \|_\infty m_T^{-1} + 1) \limsup_{r \to 0} L_f(r) < \lambda_1,$$

where m_T is defined in [1, Lemma 2.2];

then there exists $\rho > 0$ such that the inverse source problem (**IP**) admits a unique solution, provided that $\| \xi \| \leq \rho$.

We now consider the special case where the final data does not depend on the state. In this case, we have the following result concerning the Lipschitz type stability of the solution map.

Theorem 1.1. Assume that (H1), (H2), (H4) hold, with f being globally Lipschitz continuous with constant L_f. Additionally, suppose that:

(**H**3) The function $\varphi: C([0,T];L^2(\Omega)) \to \mathbb{V}_{1/2}$ is a constant mapping, that is, $\varphi(\omega) = g$ for some $g \in \mathbb{V}_{1/2}$ and all $\omega \in C([0,T];L^2(\Omega))$.

Then the solution map $(\xi, g) \mapsto (u, z)$ is Lipschitz continuous as a mapping from $L^2(\Omega) \times \mathbb{V}_{1/2}$ to $C([0,T];L^2(\Omega)) \times \mathbb{V}_{-1/2}$, provided that

$$L_f\left(\| h \|_\infty m_T^{-1} + 1\right) < \lambda_1.$$

Our next goal is to deal with the regularity of solutions to the problem (**IP**). With this goal in mind, we will show that under additional hypotheses h, f and φ take more regular values, the obtained solution to the problem (**IP**) is strong in the sense of the following definition.

Definition 1.1. A pair $(u,z) \in C([0,T];L^2(\Omega)) \times L^2(\Omega)$ is said to be a strong solution to the problem (**IP**) iff (1), (3), and (4) hold as equations in $L^2(\Omega)$.

Let $\beta = \max\left\{\frac{\alpha}{2}, \frac{1-\alpha}{2}\right\}$. In order to deal with strong solution, we further require that

($\tilde{H}$ 1) The real function h satisfies (H1) with $h \in C^\beta[0,T]$;

($\tilde{H}$ 2) The nonlinear function $f:[0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies $f(\cdot, 0) = 0$, and is locally Lipschitz-Hölder, that is, for each $r > 0$, there exists $L_f(r) > 0$ such that

$$\| f(t_1, v_1) - f(t_2, v_2) \| \le L_f(r)\left[| t_1 - t_2 |^\beta + \| v_1 - v_2 \| \right],$$

for all $t_i \in [0,T], \| v_i \|, \le r, i \in \{1,2\}$;

($\tilde{H}$ 3) The function φ satisfies (H3) with values in $\mathbb{V}_1$.

Having these assumptions in hand, we can prove the regularity of the solution for the problem (**IP**) which reads as follows:

Theorem 1.2. Let ($\tilde{H}$ 1)-($\tilde{H}$ 3), and (H4) hold. Then there exists $\rho > 0$ such that the identification problem (**IP**) has a unique strong solution on $[0,T]$, provided that $\| \xi \| \le \rho$.

2. Proof of the main results

This section is devoted to proving the main results stated in Sect. 1. We first show the proof of continuous dependence result for our problem.

Proof of Theorem 1.1.

Note that, by Corollary 3.1 in [1], for each for each $(\xi, g) \in L^2(\Omega) \times \mathbb{V}_{1/2}$, the problem (**IP**) has a unique solution (u,z), where the representations of u, z are given by

$$u(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)\big(zh(\tau) + f(\tau, u(\tau))\big)d\tau, t \in [0,T], \tag{5}$$

$$z = \mathbb{Q}\Big[g - \mathcal{S}_\alpha(T)\xi - \int_0^T \mathcal{R}_\alpha(T-\tau) f(\tau, u(\tau)) d\tau\Big]. \tag{6}$$

See Eqs. (31), (32) in [1].

Now let (u_i, z_i) be two solutions of the problem (**IP**) with respect to (ξ_i, g_i), $i \in \{1,2\}$. Then, in view of the representation (6), we find that

$$\begin{aligned}
&\| z_1 - z_2 \|_{\mathbb{V}_{-1/2}} \\
&\leq \| \mathbb{Q}(g_1 - g_2) \|_{\mathbb{V}_{-1/2}} + \| \mathbb{Q}\mathcal{S}_\alpha(T)(\xi_1 - \xi_2) \|_{\mathbb{V}_{-1/2}} \\
&\quad + \| \mathbb{Q}\mathcal{R}_\alpha * \big(f(\cdot, u_1(\cdot)) - f(\cdot, u_2(\cdot)) \big)(T) \|_{\mathbb{V}_{-1/2}} \\
&\leq m_T^{-1} \| g_1 - g_2 \|_{\mathbb{V}_{1/2}} + m_T^{-1}\lambda_1^{-1/2}(1*\ell)(T)^{-1} \| \xi_1 - \xi_2 \| + m_T^{-1}\lambda_1^{-1/2} L_f \| u_1 - u_2 \|_\infty .
\end{aligned}$$

Additionally, by the formulation (5) and Lemma 2.1 in [1], one has

$$\begin{aligned}
&\| u_1(t) - u_2(t) \| \\
&\leq s_\alpha(t, \lambda_1) \| \xi_1 - \xi_2 \| + \| h \|_\infty \lambda_1^{-1/2} \| z_1 - z_2 \|_{\mathbb{V}_{-1/2}} + \int_0^t r_\alpha(t-\tau, \lambda_1) L_f \| u_1(\tau) - u_2(\tau) \| d\tau \\
&\leq \| \xi_1 - \xi_2 \| + \| h \|_\infty \lambda_1^{-1/2} \| z_1 - z_2 \|_{\mathbb{V}_{-1/2}} + \lambda_1^{-1} L_f \| u_1 - u_2 \|_\infty \\
&\leq \| \xi_1 - \xi_2 \| + \| h \|_\infty m_T^{-1}\lambda_1^{-1} \| \xi_1 - \xi_2 \| + \| h \|_\infty m_T^{-1}\lambda_1^{-1/2} \| g_1 - g_2 \|_{\mathbb{V}_{1/2}} \\
&\quad + L_f(\| h \|_\infty m_T^{-1} + 1)\lambda_1^{-1} \| u_1 - u_2 \|_\infty, \forall t \in [0,T].
\end{aligned}$$

Consequently, we arrive at

$$\| u_1 - u_2 \|_\infty \leq \frac{1 + \| h \|_\infty m_T^{-1}\lambda_1^{-1}}{1-\kappa} \| \xi_1 - \xi_2 \| + \frac{\| h \|_\infty m_T^{-1}\lambda_1^{-1/2}}{1-\kappa} \| g_1 - g_2 \|_{\mathbb{V}_{1/2}},$$

where $\kappa := 1 - L_f(\| h \|_\infty m_T^{-1} + 1)\lambda_1^{-1}$. The proof of Theorem 1.1 is therefore complete. □

We finally prove the regularity of solutions to the problem (**IP**).

Proof of Theorem 1.2. Note that the existence result is ensured by Theorem 3.1 in [1]. Denote

$$\tilde{f}(t) = f(t, u(t)), \quad \tilde{h}(t) = zh(t) + \tilde{f}(t), t \in [0,T],$$

where (u, z) be the solution to the problem (**IP**) whose representations are given in Eqs. (5), (6).

Let us first show that the solution u is Hölder continuous on $(0,T]$. Indeed, fix ϵ, ρ, R as in the proof of Theorem 3.1 in [1]. For $t \in (0,T]$, $\theta \in (0, T-t]$, we have

$$\begin{aligned}
\| u(t+\theta) - u(t) \| &\leq \| [\mathcal{S}_\alpha(t+\theta) - \mathcal{S}_\alpha(t)]\xi \| + \| \int_0^t [\mathcal{R}_\alpha(t+\theta-\tau) - \mathcal{R}_\alpha(t-\theta)] zh(\tau) d\tau \| \\
&\quad + \| \int_t^{t+\theta} \mathcal{R}_\alpha(t+\theta-\tau) zh(\tau) d\tau \| + \| \int_t^{t+\theta} \mathcal{R}_\alpha(t+\theta-\tau) \tilde{f}(\tau) d\tau \\
&\quad + \| \int_0^t [\mathcal{R}_\alpha(t+\theta-\tau) - \mathcal{R}_\alpha(t-\theta)] \tilde{f}(\tau) d\tau \| \\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{7}$$

We now wish to estimate the five terms in (7) term by term. For J_1, with the aid of the mean value formula [2, Theorem 3.2.6, p. 119] and Lemma 2.1(i) in [1], we have

$$J_1 = \| \int_0^1 \theta \mathcal{S}_\alpha'(t+\theta\tau)\xi d\tau \| \le \theta \int_0^1 \| \mathcal{S}_\alpha'(t+\theta\tau)\xi \| d\tau \le \theta \| \xi \| \int_0^1 \frac{d\tau}{t+\theta\tau}$$
$$= \| \xi \| \ln\Big(1+\frac{\theta}{t}\Big) \le \| \xi \| \alpha^{-1} t^{-\alpha}\theta^{\alpha}, \tag{9}$$

thanks to the basic inequality $\ln(1+r) \le \alpha^{-1} r^{\alpha}$ for any $r > 0$.

By employing a similar argument to that we have used in the proof of Theorem 3.1 and Proposition 2.1(i) in [1], we have

$$J_4 \le (\alpha_f + \epsilon) R \int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_1) d\tau = (\alpha_f+\epsilon) R \int_0^\theta r_\alpha(\tau,\lambda_1) d\tau \le (\alpha_f+\epsilon) R \int_0^\theta \ell(\tau) d\tau$$
$$\le (\alpha_f+\epsilon) R \nu_1^{-1}\theta \le (\alpha_f+\epsilon) R \nu_1^{-1} T^{1-\alpha/2}\theta^{\alpha/2}. \tag{10}$$

Using the Hölder inequality and Proposition 2.1(i) in [1], one has

$$J_3^2 = \sum_{n=1}^\infty \Big(\int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_n) z_n h(\tau) d\tau\Big)^2 \le \sum_{n=1}^\infty \int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_n) d\tau \int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_n) z_n^2 h^2(\tau) d\tau$$
$$\le \| h \|_\infty^2 \sum_{n=1}^\infty \int_0^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_n) d\tau \int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_n) z_n^2 d\tau \le \| h \|_\infty^2 \sum_{n=1}^\infty \int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_n)\lambda_n^{-1} z_n^2 d\tau$$
$$\le \| h \|_\infty^2 \int_t^{t+\theta} r_\alpha(t+\theta-\tau,\lambda_1) \| z \|_{\mathbb{V}_{-1/2}}^2 d\tau \le \| h \|_\infty^2 \int_t^{t+\theta} \nu_1^{-1} \| z \|_{\mathbb{V}_{-1/2}}^2 d\tau$$
$$\le \| h \|_\infty^2 \nu_1^{-1} \| z \|_{\mathbb{V}_{-1/2}}^2 T^{1-\alpha}\theta^{\alpha}.$$

Therefore

$$J_3 \le \| h \|_\infty \nu_1^{-1/2} \| z \|_{\mathbb{V}_{-1/2}} T^{(1-\alpha)/2}\theta^{\alpha/2}. \tag{11}$$

Concerning J_2, note that

$$J_2^2 = \sum_{n=1}^\infty \bigg(\int_0^t [r_\alpha(t+\theta-\tau,\lambda_n) - r_\alpha(t-\tau,\lambda_n)] z_n f(\tau) d\tau\bigg)^2.$$

Denote $\mu_n(t,\tau) = r_\alpha(t+\theta-\tau,\lambda_n) - r_\alpha(t-\tau,\lambda_n), n = 1,2,\ldots$ Due to the differentiability of $r_\alpha(\cdot,\lambda_n)$ (see [3, Lemma 2.4(b)]), we get

$$|\mu_n(t,\tau)| \le \int_{t-\tau}^{t+\theta-\tau} | r_\alpha'(\zeta) | d\zeta \le \int_{t-\tau}^{t+\theta-\tau} (\tau^{-1} + \nu_2\nu_1^{-1}\tau^{-\alpha}) d\tau$$
$$= (1+\nu_2\nu_1^{-1})(1-\alpha)^{-1}\Big((t+\theta-\tau)^{1-\alpha} - (t-\tau)^{1-\alpha}\Big)$$
$$\le 2(1+\nu_2\nu_1^{-1})(1-\alpha)^{-1}\theta^{1-\alpha},$$

thanks to the fundamental inequality $| a^\alpha - b^\alpha | \le | a-b |^\alpha$ for all $a, b \ge 0$. On the other hand, since the function $r_\alpha(\cdot,\lambda_n)$ is nonincreasing, we also have

$$|\mu_n(t,\tau)| = r_\alpha(t-\tau,\lambda_n) - r_\alpha(t+\theta-\tau,\lambda_n) \le 2 r_\alpha(t-\tau,\lambda_n).$$

Therefore

$$\Big(\int_0^t \mu_n(t,\tau) z_n h(\tau) d\tau\Big)^2 \le \int_0^t |\mu_n(t,\tau)|\, d\tau \int_0^t |\mu_n(t,\tau)|\, z_n^2 h^2(\tau) d\tau$$
$$\le 4(1+\nu_2\nu_1^{-1})(1-\alpha)^{-1}\theta^{1-\alpha}\int_0^t d\tau \int_0^t r_\alpha(t-\tau,\lambda_n) z_n^2 h^2(\tau) d\tau$$
$$\le 4T(1+\nu_2\nu_1^{-1})(1-\alpha)^{-1}\theta^{1-\alpha}\lambda_n^{-1} z_n^2 \,\|h\|_\infty^2 .$$

By this, we obtain

$$J_2 \le 2T^{1/2}(1+\nu_2\nu_1^{-1})^{1/2}(1-\alpha)^{-1/2}\theta^{(1-\alpha)/2}\,\|z\|_{\mathbb{V}_{-1/2}}\|h\|_\infty . \tag{12}$$

With respect to J_5, observe that

$$\big\|\big[\mathcal{R}_\alpha(t+\theta-\tau)-\mathcal{R}_\alpha(t-\tau)\big]\tilde f(\tau)\big\|$$
$$= \theta \,\Big\|\int_0^1 \mathcal{R}_\alpha'(t+\theta\zeta-\tau)\tilde f(\tau) d\zeta\Big\|$$
$$\le (\alpha_f+\epsilon)R\theta\int_0^1 \big[\nu_1^{-1}(t+\theta\zeta-\tau)^{-1} + \nu_1^{-2}\nu_2\Gamma(1-\alpha)^{-1}(t+\theta\zeta-\tau)^{-\alpha}\big] d\theta$$
$$= (\alpha_f+\epsilon)R\Big[\nu_1^{-1}\ln\Big(1+\frac{\theta}{t-\tau}\Big) + \nu_1^{-2}\nu_2\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}\Big((t+\theta-\tau)^{1-\alpha}-(t-\tau)^{1-\alpha}\Big)\Big]$$
$$\le (\alpha_f+\epsilon)R\big[\nu_1^{-1}\alpha^{-1}(t-\tau)^{-\alpha}\theta^{\alpha} + \nu_1^{-2}\nu_2\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}\theta^{1-\alpha}\big],$$

thanks to the regularity of $\mathcal{R}_\alpha(\cdot)$ obtained in Lemma 2.1(ii) in [1]. Integrating the last inequality on $[0,t]$, we find that

$$J_5 \le \nu_1^{-1}(1-\alpha)^{-1}(\alpha_f+\epsilon)R\big[\alpha^{-1}t^{1-\alpha}\theta^{\alpha} + \nu_1^{-1}\nu_2\Gamma(1-\alpha)^{-1}\theta^{1-\alpha}t\big]$$
$$\le \nu_1^{-1}(1-\alpha)^{-1}(\alpha_f+\epsilon)R\big[\alpha^{-1}T^{1-\alpha}\theta^{\alpha} + \nu_1^{-1}\nu_2\Gamma(1-\alpha)^{-1}\theta^{1-\alpha}T\big]. \tag{13}$$

Thus, we obtain from (8)–(12) and (7) the following bound

$$\|u(t+\theta)-u(t)\| \le C_\beta\theta^{\beta},$$

where

$$C_\beta = \|\xi\|\alpha^{-1}t^{-\alpha}T^{\beta} + (\alpha_f+\epsilon)T\Big[\nu_1^{-1}T^{-\alpha/2} + \nu_1^{-1}(1-\alpha)^{-1}\Big(\alpha^{-1}T^{-\alpha+\beta} + \nu_1^{-1}\nu_2\Gamma(1-\alpha)^{-1}\times T^{\beta}\Big)\Big]R$$
$$+\|h\|_\infty T^{1/2}\big[\nu_1^{-1/2}T^{-\alpha/2} + 2(1+\nu_2\nu_1^{-1})^{1/2}(1-\alpha)^{-1/2}\big]\|z\|_{\mathbb{V}_{-1/2}} .$$

Therefore, due to the assumption ($\tilde H$ 2), it holds that

$$\|\tilde f(t+\theta)-\tilde f(t)\| = \|f(t+\theta,u(t+\theta)) - f(t,u(t))\|$$
$$\le (\alpha_f+\epsilon)R[\theta^{\beta} + \|u(t+\theta)-u(t)\|$$
$$\le (\alpha_f+\epsilon)(1+C_\beta)R\theta^{\beta}.$$

Making use of this inequality and following the process of [3, Theorem 4.2], it implies that $\Delta(\mathcal{R}_\alpha * \tilde f)(T)\in L^2(\Omega)$. Thus, thanks to [3, Lemma 2.4(i)], $\mathbb{Q}(\mathcal{R}_\alpha * \tilde f)(T)\in L^2(\Omega)$. In addition, by our condition on φ, one gets $\varphi(u)-\mathcal{S}_\alpha(T)\xi$ belonging to $\mathbb{V}_1$, thanks to Lemma 2.2(i) in [1]. Based on the above considerations, Lemma 2.2(ii) in [1], and the decomposition

$$z = \mathbb{Q}[\varphi(u)-\mathcal{S}_\alpha(T)\xi] - \mathbb{Q}(\mathcal{R}_\alpha * \tilde f)(T),$$

it thus follows that $z \in L^2(\Omega)$.

Now denote C_h the Hölder constant of h. In order to finish the proof of this theorem, following [3, Theorem 4.2], it remains to show that the function $\tilde{f}$ is Hölder continuous. Indeed, from the estimates obtained above, we have that

$$\begin{aligned} \| \tilde{h}(t+\theta) - \tilde{h}(t) \| &\le \| z \| | h(t+\theta) - h(t) | + \| \tilde{f}(t+\theta) - \tilde{f}(t) \| \\ &\le C_h \| z \| \theta^\beta + (\alpha_f + \epsilon)(1 + C_\beta) R \theta^\beta \\ &= [C_h \| z \| + (\alpha_f + \epsilon)(1 + C_\beta) R] \theta^\beta, \end{aligned}$$

which completes the proof. □

Before closing this section, let us mention that if two nonlinearity functions f, φ fulfill the global Lipschitz conditions (H2)$'$, (H3)$'$:

(H2)$'$ $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ *is globally Lipschitz continuous, that is, there exists* $L_f > 0$ *such that*

$$\| f(t, v_1) - f(t, v_2) \| \le L_f \| v_1 - v_2 \|, \forall t \in [0,T], v_1, v_2 \in L^2(\Omega),$$

(H3)$'$ $\varphi : C([0,T]; L^2(\Omega)) \to \mathbb{V}_{1/2}$ *is globally Lipschitz continuous, that is, there exists* $L_\varphi > 0$ *such that*

$$\| \varphi(w_1) - \varphi(w_2) \|_{\mathbb{V}_{1/2}} \le L_\varphi \| w_1 - w_2 \|_\infty, \forall w_1, w_2 \in C([0,T]; L^2(\Omega)),$$

then the conclusion of Theorem 1.2 remains true. In this case, the existence of the solution for the problem (**IP**) and its regularity are proved by using the same arguments as done in the proof of Corollary 3.1 in [1], Theorem 1.2, respectively. To be more precise, we have the following corollary.

Corollary 5. Let ($\tilde{H}$ 1), (H2)$'$, and (H3)$'$ be satisfied. Then the problem (**IP**) has a unique strong solution.

3. Conclusion and dicussion

In this study, we investigate the source identification problem for recovering a spatially varying parameter in the right-hand side of a nonlinear fractional mobile-immobile equation. By imposing appropriate conditions on the model parameters, we establish the data dependence of solutions (Theorem 1.1). Furthermore, under sufficient regularity assumptions on the final data and perturbation terms, we demonstrate that the obtained solution is strong (Theorem 1.2 and Corollary 5).

Several meaningful questions of a similar nature arise, such as:

(i) determining spatially varying parameters in anomalous diffusion equations [4]–[8], Rayleigh-Stokes equations [9], [10], or sub-diffusion equations [7], [11], [12], and other fractional models [13]–[21] using additional measurements;

(ii) inferring the relationship between input data and the regularity of solutions.

Below, we propose a partial list of open problems that can be formulated explicitly or implicitly for future research:

1. Investigate the existence, data dependence, and regularity of solutions for the inverse problem (**IP**) using nonlocal observations and perturbed nonlinearities of polynomial, gradient, or advective types.
2. Derive sufficient conditions to ensure regularity in spatial variables under weaker assumptions.

3. Establish the existence, stability, and convergence of numerical solutions for such nonlinear fractional models.

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