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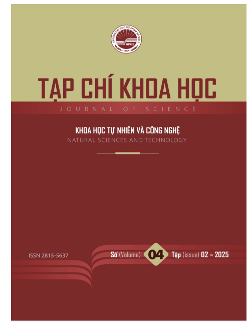

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Final value problem for fractional reaction-subdiffusion equations

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Abstract

We investigate the existence of a mild solution to the final value problem for a class of fractional reaction-subdiffusion nonlinear equations, where the nonlinearity may take weak values. We want to demonstrate the unique existence of a mild solution by using the Banach fixed-point theorem. In order to do this, we construct some new estimates for the resolvent function and the resolvent operator, based on the existing resolvent theory. From our point of view, the nonlinearity, which takes values in Hilbert scales, presents some technical difficulties but allows us to examine broader classes of problems, since by which it can contain a polynomial or gradient term arising from various physical circumstances.

Keywords: Reaction-subdiffusion equations, nonlocal PDE, final value problem, resolvent theory, Hilbert scales

1. Introduction

Given a domain $\Omega \subset \mathbb{R}^d$, assume that Ω is a bounded domain whose the boundary $\partial\Omega$ is smooth. Our problem has the following form:

$$\partial_t u - \partial_t^{1-\alpha}(\nu_1 \Delta u - \nu_2 u) = f(u) \text{ in } \Omega,\ t \in (0,T), \quad (1)$$

$$u = 0 \text{ on } \partial\Omega,\ t > 0, \quad (2)$$

$$u(T,\cdot) = \xi \text{ in } \Omega, \quad (3)$$

here $u(t,x)$ is a function defined on $(0,T] \times \Omega$, the function f can be chosen willingly, ν_1, ν_2 are positive parameters, $\partial_t = \frac{\partial}{\partial t}$, $\partial_t^{1-\alpha}$ is the nonlocal derivative operator of Riemann-Liouville type, that means

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$$\partial_t^{1-\alpha} v(t) = \frac{d}{dt}\int_0^t g_\alpha(t-s)v(s)ds = \partial_t\big(g_\alpha * v(t)\big), t > 0$$

here $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \in L^1_{loc}(\mathbb{R}^+)$, and the symbol "$*$" is used to denote the Laplace convolution.

In recent years, nonlocal partial differential equations, in particular, fractional partial differential equations have received attention from mathematicians, thanks to the application of these equations in modeling real-life processes. Among many branches of this research area, we focus on studying the reaction-subdiffusion equations (see [1]–[3] for additional information of reaction-subdiffusion equations in physics and chemistry). The subdiffusion process is a special case of anomalous diffusion, characterized by a nonlinear relationship between the mean squared displacement and time. Anomalous diffusion plays a crucial role in physics, such as the diffusion of proteins within cells or diffusion through porous materials, where conventional diffusion equations are ineffective for modeling these phenomena. Many studies have demonstrated the advantages of fractional models over classical models in describing anomalous diffusion processes, where the fractional derivative can be considered with respect to both time and space variables. The fractional derivative with respect to the spatial variable can be used to describe the spreading of particles with a rate which is inconsistent compared to classical Brownian motion, this diffusive motion is known as Lévy walk and can be simulated using the Riesz-Feller fractional derivative with respect to the spatial variable, as specifically addressed in the work [4]. In the paper [5], the authors study the long and short time behavior of the solutions to a class of non-local in time subdiffusion equations, using tools of the theory of Volterra equations. When examining ecological models in biology, the problem of determining the population density of a certain species is posed in [6], with a non-local diffusion component and a nonlinear reaction component. Some works on numerical solutions for the reaction-subdiffusion equation can be found in the literature [7]–[10].

One efficient tool used to describe anomalous diffusion processes is fractional calculus. We consider a class of reaction-diffusion equations as follows

$$\partial_t u(t,x) - \partial_t \int_0^t g_\alpha(t-s)Au(s,x)ds = f, t > 0, x \in \Omega, \quad \text{(R-D)}$$

where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha \in (0,1), t > 0$ and A is an elliptic operator.

The existence and finite-time blow-up of solutions for the class of equations (R-D) have been studied in [11], [12]. The existence, uniqueness of solutions, and stability, as well as the regularity of solutions for the parameter identification problem with the reaction-diffusion equation of the form (R-D), where $A = \nu_1\Delta - \nu_2, \nu_1 > 0, \nu_2 > 0$, have been studied in the paper [13].

For a class of reaction-diffusion equations of the form (R-D) considered above, the external force f can be considered either linear or nonlinear. In the case where $f = f(u)$ is a nonlinear function, it is often assumed to take values in the space $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^d$ in order to utilize many useful tools. However, this assumption has its limitations, since f may be difficult to express in polynomial form such as $f(u) = |u|^p, p > 1$ or to include gradient components such as $f(u) = h(x) \cdot \nabla u$, where h is a function of the spatial variable $x \in \Omega$. To overcome this limitations, one can assume that $f(u)$ belongs to Hilbert scales of nonpositive order and make use of the relationship between Hilbert scales and fractional Sobolev spaces (see [14]–[16], [19]–[20]). On the other hand, the final value problem arises when there are no observations at the initial time, and in that case, we need current observations to detect the previous states of the system. This problem emerges from many practical applications in signal processing, image restoration, medical diagnosis, etc (see [17], [19], [20]).

In this paper, we study the existence of a mild solution to the final value problem for a class of fractional reaction-subdiffusion nonlinear equations, where the nonlinearity may take values in Hilbert scales of nonpositive order. We first establish some estimates for the resolvent operator in Hilbert scales and this enables us to demonstrate that the solution operator admits a unique fixed point in a suitable space by using the Banach fixed point theorem. Additional information of the techniques used in this paper can be found in [18]–[20].

2. Estimates for resolvent operator

We want to emphasize that the nonlinearity f can take weak values, in other words, f belongs to a Hilbert scale of nonpositive order. To recall the notion of Hilbert scales, we consider the Laplace operator subject to the homogeneous Dirichlet boundary condition denoted by the symbol $-\Delta$. Then $-\Delta$ can be represented as follows:

$$-\Delta = \sum_{n=1}^{+\infty} \lambda_n (\cdot, e_n) e_n,$$

where $\{(e_n, \lambda_n)\}$ is the eigensystem of $-\Delta$ satisfying that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$ as $n \to +\infty$, and we also recall that $\{e_n\}$ is an orthonormal basis of $L^2(\Omega)$. Here the notation $(\cdot,\cdot)$ stands for the scalar product in $L^2(\Omega)$.

For $\varrho \geq 0$, consider the function space

$$\mathbb{H}^{\varrho} := \left\{ \varphi \in L^2(\Omega) \mid \| \varphi \|_{\varrho}^2 := \sum_{n=1}^{+\infty} \lambda_n^{\varrho} (\varphi, e_n)^2 < +\infty \right\}$$

and its dual space $\mathbb{H}^{-\varrho}$ and the duality pairing $\langle \cdot,\cdot \rangle_{-\varrho,\varrho}$ on $\mathbb{H}^{-\varrho} \times \mathbb{H}^{\varrho}$. The space $\mathbb{H}^{-\varrho}$ can be equipped with the norm

$$\| \varphi \|_{-\varrho}^2 := \sum_{n=1}^{+\infty} \left| \lambda_n^{-\frac{\varrho}{2}} \langle \varphi, e_n \rangle_{-\varrho,\varrho} \right|^2 < +\infty.$$

We can see that $\mathbb{H}^{\varrho_2} \hookrightarrow \mathbb{H}^{\varrho_1}$ and $\mathbb{H}^{-\varrho_1} \hookrightarrow \mathbb{H}^{-\varrho_2}$ when $\varrho_2 \geq \varrho_1 \geq 0$. The set of all Hilbert space $\mathbb{H}^{\varrho}$, $\varrho \in \mathbb{R}$ is called the Hilbert scales. So if f belongs to a Hilbert scale of nonpositive order then f can be chosen in a space which can be larger than $L^2(\Omega)$.

For the formulation of mild solution, we consider the following relaxation problem:

$$\omega'(t) + (\nu_1 \lambda + \nu_2)(g_\alpha * \omega)'(t) = 0, t > 0 \tag{4}$$

$$\omega(0) = 1, \tag{5}$$

where ω is a scalar function (we call it *resolvent function*), λ, ν_1, ν_2 are positive parameters. Integrating both sides of (4) we have

$$\omega(t) + (\nu_1 \lambda + \nu_2)(g_\alpha * \omega)(t) = 1. \tag{6}$$

We now find the representations and estimates of the resolvent function by using the properties of the Mittag-Leffler functions. The Mittag-Leffler function $E_{\alpha,\beta}$ is given by:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z \in \mathbb{C}, \alpha, \beta > 0.$$

By using the Laplace transform, one has:

$$s(t,\lambda) := E_{\alpha,1}(-\lambda t^\alpha); r(t,\lambda) := t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \tag{7}$$

where λ is positive parameter, are solutions of the following Volterra integral equations

$$s(t,\lambda)+\lambda\int_0^t g_\alpha(t-\tau)s(t,\tau)d\tau=1,\ t\geq 0, \tag{8}$$

$$r(t,\lambda)+\lambda\int_0^t g_\alpha(t-\tau)r(t,\tau)d\tau=g_\alpha(t),\ t>0. \tag{9}$$

We can easily see that (6) is identical to (8) with $\omega(t)$ is replaced by $s(t,\lambda)$, λ is replaced by $\nu_1\lambda+\nu_2$, and the Laplace convolution is written in integral form. Note that when $\lambda>0$ then the function $s(t,\lambda)$ is completely monotonic on $(0,\infty)$, that means $(-1)^n\frac{\partial^n}{\partial t^n}s(t,\lambda)\geq 0$ for all $n=0,1,2,\cdots$ and $t>0$. Some properties of $s(t,\lambda)$ are listed in the following proposition (see [13]):

Proposition 2.1. *Suppose that $s(t,\lambda)$ is a solution of (8). Then*

(a) We have the lower and upper bound for $s(t,\lambda)$ as follows:

$$\frac{1}{1+\lambda\Gamma(1-\alpha)t^\alpha}\leq s(t,\lambda)\leq\frac{1}{1+\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}},\ \forall t\geq 0,\ \lambda>0.$$

(b) For each $t>0$, the function $\lambda\mapsto s(t,\lambda)$ and the function $t\mapsto s(t,\lambda)$ are nonincreasing.

(c) The function $v(t)=s(t,\lambda)v_0+\int_0^t s(t-\tau,\lambda)h(\tau)d\tau$ is the solution of

$$\begin{aligned} v'(t)+\lambda(g_\alpha*v)'(t) &= h(t),\\ v(0) &= v_0. \end{aligned}$$

By applying Proposition 2.1 for ω, we obtain the following estimate:

$$\frac{1}{1+(\nu_1\lambda+\nu_2)\Gamma(1-\alpha)t^\alpha}\ \leq\ \omega(t,\nu_1\lambda+\nu_2)\ \leq\ \frac{1}{1+\frac{(\nu_1\lambda+\nu_2)t^\alpha}{\Gamma(1+\alpha)}},\ \forall t>0,\ \lambda>0. \tag{10}$$

In the next step, we will find the solution representation for the linear initial value problem (consider $F=F(t,x)$ for simplification):

$$\partial_t u-\partial_t^{1-\alpha}(\nu_1\Delta u-\nu_2 u)=F\ \text{ in }\Omega, t\in(0,T], \tag{11}$$

$$u=0\ \text{ on }\partial\Omega,\ t\in[0,T], \tag{12}$$

$$u(0)=\eta\ \text{ in }\Omega, \tag{13}$$

where $F\in C\big([0,T];L^2(\Omega)\big)$.

Since $\{e_n\}$ is an orthonormal basis of $L^2(\Omega)$ (we have mentioned above), we can assume that

$$u(t)=\sum_{n=1}^{\infty}u_n(t)e_n,\ F(t)=\sum_{n=1}^{\infty}F_n(t)e_n.$$

Putting that into (11), one gets

$$\begin{aligned} &u_n{}'(t)+(\nu_1\lambda_n+\nu_2)(g_\alpha*u_n)'(t)=F_n(t),\\ &u_n(0)=\eta_n:=(\eta,e_n). \end{aligned}$$

Using the Proposition 2.1(c), we have

$$u_n(t)=\omega(t,\nu_1\lambda_n+\nu_2)\eta_n+\omega(\cdot,\nu_1\lambda_n+\nu_2)*F_n(t).$$

Therefore

$$u(t)=S(t)\eta+S*F(t), \tag{14}$$

where $S(t)$ is the *resolvent operator* given by

$$S(t)\eta=\sum_{n=1}^{\infty}\omega(t,\nu_1\lambda_n+\nu_2)\eta_n e_n,\ \eta\in L^2(\Omega). \tag{15}$$

We can see that $S(t)$ is a bounded linear operator on $L^2(\Omega)$ for all $t \geq 0$. Finally, we construct the solution formula for the final value problem in the linear case:

$$\partial_t u - \partial_t^{1-\alpha}(\nu_1 \Delta u - \nu_2 u) = F \quad \text{in } \Omega, t \in (0,T], \tag{16}$$

$$u = 0 \quad \text{on } \partial\Omega,\ t \in (0,T], \tag{17}$$

$$u(T,\cdot) = \xi \quad \text{in } \Omega, \tag{18}$$

here $F \in C\big([0,T]; L^2(\Omega)\big)$. From (13), let $t = T$ we have:

$$\xi = S(T)\eta + \int_0^T S(T-\tau)F(\tau)d\tau.$$

Then

$$\eta = S(T)^{-1}\left[\xi - \int_0^T S(T-\tau)F(\tau)d\tau\right].$$

Therefore, the solution of (16)-(18) is given by

$$u(t) = P(t)[\xi - S * F(T)] + S * F(t), \tag{19}$$

here

$$P(t) = S(t)S(T)^{-1} = \sum_{n=1}^{\infty} \frac{\omega(t,\nu_1\lambda_n + \nu_2)}{\omega(T,\nu_1\lambda_n + \nu_2)}(\cdot, e_n)e_n. \tag{20}$$

The properties of S and P can be shown in the following lemmas.

Lemma 2.1. *Suppose that* $\{S(t)\}_{t\geq 0}$ *is a family of resolvent operators given by (15),* $v \in L^2(\Omega)$ *and* $T > 0$. *Then for* $\gamma, \gamma' \in (0,1)$, $\mu > 0$ *and* $g \in C([0,T]; \mathbb{H}^{\mu - 2\gamma\gamma'})$, *we have*

$\| S * g(t) \|_\mu^2 \leq M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^t (t-\tau)^{-\alpha\gamma'} \| g(\tau) \|_{\mu-2\gamma\gamma}^2 \ d\tau$, *where* $M_1 = \big(\Gamma(1+\alpha)\big)^{2\gamma'} \nu_2^{2\gamma} \nu_1^{-2\gamma\gamma'}$.

Proof. Assume that $g \in C([0,T]; \mathbb{H}^{\mu-2\gamma\gamma})$. Then

$$\| S * g(t) \|_\mu^2 = \sum_{n=1}^{\infty} \lambda_n^\mu \left(\int_0^t \omega(t-\tau, \nu_1\lambda_n + \nu_2) g_n(\tau) d\tau\right)^2,\ g_n(\tau) = (g(\tau), e_n).$$

By using the Hölder inequality we have

$$\left(\int_0^t \omega(t-\tau, \nu_1\lambda_n + \nu_2) g_n(\tau) d\tau\right)^2$$
$$\leq \left(\int_0^t \omega(t-\tau, \nu_1\lambda_n + \nu_2) d\tau\right)\left(\int_0^t \omega(t-\tau, \nu_1\lambda_n + \nu_2)|g_n(\tau)|^2 d\tau\right).$$

From (10) we have the estimate:

$$\begin{aligned}\omega(t-\tau,\nu_1\lambda_n+\nu_2) &\le \frac{1}{1+\dfrac{\nu_2\left(\dfrac{\nu_1}{\nu_2}\lambda_n+1\right)(t-\tau)^\alpha}{\Gamma(1+\alpha)}}\\
&\le \frac{1}{1+\dfrac{\nu_2\left(\dfrac{\nu_1^\gamma}{\nu_2^\gamma}\lambda_n^\gamma\right)(t-\tau)^\alpha}{\Gamma(1+\alpha)}}\\
&\le \frac{\left(\Gamma(1+\alpha)\right)^{\gamma'}}{\nu_2^{\gamma'}\dfrac{\nu_1^{\gamma\gamma'}}{\nu_2^{\gamma\gamma'}}\lambda_n^{\gamma\gamma'}(t-\tau)^{\alpha\gamma'}}\\
&= \frac{\left(\Gamma(1+\alpha)\right)^{\gamma'}}{\nu_2^{(1-\gamma)\gamma'}\nu_1^{\gamma\gamma}\lambda_n^{\gamma\gamma'}(t-\tau)^{\alpha\gamma'}},\end{aligned}$$

here we use the inequality $1+b\ge b^\kappa$ with $b>0,\kappa\in(0,1)$, and κ take values $\gamma,\gamma'\in(0,1)$.

Then

$$\begin{aligned}\int_0^t\omega\,(t-\tau,\nu_1\lambda_n+\nu_2)d\tau &\le \frac{\left(\Gamma(1+\alpha)\right)^{\gamma'}}{\nu_2^{(1-\gamma)\gamma'}\nu_1^{\gamma\gamma'}\lambda_n^{\gamma\gamma'}}\int_0^t\frac{1}{(t-\tau)^{\alpha\gamma'}}d\tau\\
&= \frac{\left(\Gamma(1+\alpha)\right)^{\gamma'}}{\nu_2^{(1-\gamma)\gamma'}\nu_1^{\gamma\gamma'}\lambda_n^{\gamma\gamma'}}\frac{-(t-\tau)^{1-\alpha\gamma'}}{1-\alpha\gamma'}\Big|_0^t\\
&= \frac{\left(\Gamma(1+\alpha)\right)^{\gamma'}}{\nu_2^{(1-\gamma)\gamma'}\nu_1^{\gamma\gamma'}\lambda_n^{\gamma\gamma'}}\frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'}.\end{aligned}$$

In the other hand, we have

$$\int_0^t\omega\,(t-\tau,\nu_1\lambda_n+\nu_2)|g_n(\tau)|^2d\tau \quad\le\int_0^t\frac{|g_n(\tau)|^2\left(\Gamma(1+\alpha)\right)^{\gamma'}}{\nu_2^{(1-\gamma)\gamma'}\nu_1^{\gamma\gamma'}\lambda_n^{\gamma\gamma'}(t-\tau)^{\alpha\gamma'}}d\tau.$$

So

$$\begin{aligned}&\left(\int_0^t\omega\,(t-\tau,\nu_1\lambda_n+\nu_2)g_n(\tau)d\tau\right)^2\\
&\le\lambda_n^{-2\gamma\gamma'}\left(\left(\Gamma(1+\alpha)\right)^{2\gamma'}\nu_2^{2(\gamma-1)\gamma'}\nu_1^{-2\gamma\gamma}\right)\frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'}\int_0^t\frac{|g_n(\tau)|^2}{(t-\tau)^{\alpha\gamma'}}d\tau.\end{aligned}$$

Set $\left(\Gamma(1+\alpha)\right)^{2\gamma'}\nu_2^{2(\gamma-1)\gamma'}\nu_1^{-2\gamma\gamma'}=M_1$, we obtain

$$\begin{aligned}\parallel S*g(t)\parallel_\mu^2 &\le\sum_{n=1}^\infty\lambda_n^{\mu-2\gamma\gamma'}M_1\frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'}\int_0^t\frac{|g_n(\tau)|^2}{(t-\tau)^{\alpha\gamma}}d\tau\\
&=M_1\frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'}\int_0^t(t-\tau)^{-\alpha\gamma'}\parallel g(\tau)\parallel_{\mu-2\gamma\gamma}^2\ d\tau.\end{aligned}$$

The proof is complete.

Lemma 2.2. *A family of operators* $\{P(t)\}$ *given by (20) has the following properties:*

(a) If $\gamma,\gamma'\in(0,1)$, $\mu>0$ and $\xi\in\mathbb{H}^{\mu+2(1-\gamma\gamma')}$ then $\parallel P(t)\xi\parallel_\mu^2\le t^{-2\alpha\gamma'}M_0\parallel\xi\parallel_{\mu+2(1-\gamma\gamma')}^2$. Here

$$M_0 = \left(\left(\frac{1}{\lambda_1} + \left(\nu_1 + \frac{\nu_2}{\lambda_1}\right)\Gamma(1-\alpha)T^\alpha\right)\left(\Gamma(1+\alpha)\right)^{\gamma'}\nu_2^\gamma \nu_1^{-\gamma\gamma'}\right)^2.$$

(b) For $\gamma, \gamma' \in (0,1)$, $\mu > 0$ and $g \in C([0,T]; \mathbb{H}^{\mu+2-4\gamma\gamma'})$,

$$\| P(t)[S * g(T)] \|_\mu^2 \le M_0 M_1 t^{-2\alpha\gamma'} \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^T (T-\tau)^{-\alpha\gamma'} \| g(\tau) \|_{\mu+2-4\gamma\gamma'}^2 \, d\tau.$$

Proof. (a) We consider

$$\| P(t)\xi \|_\mu^2 = \sum_{n=1}^\infty \lambda_n^\mu \left(\frac{\omega(t,\nu_1\lambda_n+\nu_2)}{\omega(T,\nu_1\lambda_n+\nu_2)}\right)^2 \xi_n^2,\ \xi_n = (\xi, e_n). \tag{21}$$

We first obtain the following estimate:

$$\begin{aligned}
\frac{\omega(t,\nu_1\lambda_n+\nu_2)}{\omega(T,\nu_1\lambda_n+\nu_2)} &\le \frac{1+(\nu_1\lambda_n+\nu_2)\Gamma(1-\alpha)T^\alpha}{1+\frac{(\nu_1\lambda_n+\nu_2)t^\alpha}{\Gamma(1+\alpha)}} \\
&\le \lambda_n \frac{\frac{1}{\lambda_n} + \left(\nu_1 + \frac{\nu_2}{\lambda_n}\right)\Gamma(1-\alpha)T^\alpha}{\frac{(\nu_1\lambda_n)^{\gamma\gamma'}\nu_2^{-\gamma}t^{\alpha\gamma}}{\left(\Gamma(1+\alpha)\right)^{\gamma'}}} \\
&\le \lambda_n \frac{\frac{1}{\lambda_1} + \left(\nu_1 + \frac{\nu_2}{\lambda_1}\right)\Gamma(1-\alpha)T^\alpha}{\frac{(\nu_1\lambda_n)^{\gamma\gamma'}\nu_2^{-\gamma}t^{\alpha\gamma'}}{\left(\Gamma(1+\alpha)\right)^{\gamma'}}} \\
&= \lambda_n^{1-\gamma\gamma'}\left(\frac{1}{\lambda_1} + \left(\nu_1 + \frac{\nu_2}{\lambda_1}\right)\Gamma(1-\alpha)T^\alpha\right) \\
&\quad \times \left(\Gamma(1+\alpha)\right)^{\gamma'}\nu_2^\gamma\nu_1^{-\gamma\gamma'}t^{-\alpha\gamma'},
\end{aligned}$$

here we use (10) and the inequality $1+b \ge b^\kappa$ with $b > 0$, $\kappa \in (0,1)$. Set

$$\left(\left(\frac{1}{\lambda_1} + \left(\nu_1 + \frac{\nu_2}{\lambda_1}\right)\Gamma(1-\alpha)T^\alpha\right)\left(\Gamma(1+\alpha)\right)^{\gamma'}\nu_2^\gamma\nu_1^{-\gamma\gamma'}\right)^2 = M_0,$$

substitute into (21), we get

$$\begin{aligned}
\| P(t)\xi \|_\mu^2 &\le t^{-2\alpha\gamma'} M_0 \sum_{n=1}^\infty \lambda_n^{\mu+2(1-\gamma\gamma')}\xi_n^2 \\
&= t^{-2\alpha\gamma'} M_0 \| \xi \|_{\mu+2(1-\gamma\gamma')}^2.
\end{aligned}$$

(b) By applying the estimate in (a) and Lemma 2.1, we obtain

$$\begin{aligned}
\| P(t)[S * g(T)] \|_\mu^2 &\le t^{-2\alpha\gamma'} M_0 \| S * g(T) \|_{\mu+2(1-\nu)}^2 \\
&\le M_0 M_1 t^{-2\alpha\gamma'} \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^T (T-\tau)^{-\alpha\gamma'} \| g(\tau) \|_{\mu+2-4\gamma\gamma'}^2 \, d\tau.
\end{aligned}$$

The proof is complete.

3. Existence of mild solution

To deal with (1)-(3), we need the following hypotheses **(F)** for the nonlinearity f:

(F) The function $f: \mathbb{H}^\mu \to \mathbb{H}^{-\theta}$ satisfies $f(0) = 0$, here $\mu > 0$ and θ is nonnegative. Moreover, there exist nonnegative functions L_f and ℓ_f such that for $u, v \in \mathbb{H}^\mu$ we have

$$\| f(u) - f(v) \|_{-\theta} \leq L_f(\| u \|_\mu, \| v \|_\mu) \| u - v \|_\mu$$

and

$$L_f(\psi\rho, \psi\rho') \geq \ell_f(\psi) L_f(\rho, \rho'), \text{ for all } \psi, \rho, \rho' > 0.$$

Based on (19), we deliver the definition of solution for (1)-(3) as follows:

Definition 3.1. *Let $\mu > 0$. A function $u \in C((0,T]; \mathbb{H}^\mu)$ is called a mild solution of (1)-(3) if*

$$u(t) = P(t)\xi - P(t)\int_0^T S(T-\tau) f(u(\tau)) d\tau + \int_0^t S(t-\tau) f(u(\tau)) d\tau, \ 0 < t \leq T.$$

We look for the solution of (1)-(3) in the function space

$$\mathbb{V}^{\mu,\alpha\gamma'} = \{u \in C((0,T]; \mathbb{H}^\mu) : u(T,\cdot) = \xi \text{ and } \| u \|_{\mu,\alpha\gamma'} := \sup_{t>0} t^{\alpha\gamma} \| u(t) \|_\mu < +\infty\},$$

here μ, α, γ' are positive and ξ is a given final data.

Theorem 3.2. *Let $\mu \in (0,1]$, $0 < \gamma, \gamma' < 1$ such that $\mu \leq 4\gamma\gamma' - 2$. Assume that **(F)** holds with $\theta = 4\gamma\gamma' - 2 - \mu$. Then there exist $\rho^* > 0$ and $\beta > 0$ such that if $\| \xi \|_{\mu+2(1-\gamma\gamma')} \leq \beta$ and*

$$6{L_f^*}^2 \left[M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \Lambda(T) + \lambda_1^{2\gamma\gamma'-} \ M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \sup_{t\in(0,T]} t^{2\alpha\gamma'} \Lambda(t) \right] < 1 \quad \textit{here}$$

$$L_f^* = \limsup_{\rho,\rho'\to 0} L_f(\rho, \rho'),$$

$$\Lambda(t) = \int_0^t (t-\tau)^{-\alpha\gamma'} \tau^{-2\alpha\gamma'} \ell_f(\tau^{\alpha\gamma})^{-2} d\tau,$$

then the problem (1)-(3) has a unique mild solution u in $\mathbb{V}^{\mu,\alpha\gamma'}$ satisfying $\| u \|_{\mu,\alpha\gamma'} \leq \rho^*$.

Proof. We consider the solution operator

$$\Phi(u)(t) = P(t)\xi - P(t)\int_0^T S(T-\tau) f(u(\tau)) d\tau + \int_0^t S(t-\tau) f(u(\tau)) d\tau, \ 0 < t \leq T.$$

We can see that $\Phi(u)(T) = \xi$, since $P(T) = I$ (here I is the identity operator). In the first step, we want to find $\rho^* > 0$ such that $\Phi(B_{\rho^*}) \subset B_{\rho^*}$, where B_{ρ^*} is a closed ball in $\mathbb{V}^{\mu,\alpha\gamma'}$ centered at 0 with radius ρ^*. Choose $\zeta > 0$ such that the following condition holds:

$$6\left({L_f^*}^2 + \zeta\right) \left[M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \Lambda(T) + \lambda_1^{2\gamma\gamma'-} \ M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \sup_{t\in(0,T]} t^{2\alpha\gamma'} \Lambda(t) \right] \leq 1.$$

From the definition of L_f^*, one can find $\rho^* > 0$ so that

$$\left(L_f(\rho, \rho')\right)^2 \leq {L_f^*}^2 + \zeta, \text{ for all } \rho, \rho' \leq \rho^*.$$

For $u \in B_{\rho^*}$, we estimate the solution operator as follows:

$$\begin{aligned} \| \Phi(u)(t) \|_\mu^2 &\leq 3 \| P(t)\xi \|_\mu^2 + 3 \| P(t)[S * f(u)(T)] \|_\mu^2 + 3 \| S * f(u)(t) \|_\mu^2 \\ &= 3[I_1(t) + I_2(t) + I_3(t)]. \end{aligned}$$

By applying Lemma 2.2, we obtain the estimate:

$$I_1(t) = \| P(t)\xi \|_\mu^2 \leq t^{-2\alpha\gamma'} M_0 \| \xi \|_{\mu+2(1-\gamma\gamma')}^2. \quad (22)$$

In addition,

$$\begin{aligned} I_2 = \|P(t)[S * f(u)(T)]\|_\mu^2 &\leq M_0 M_1 t^{-2\alpha\gamma'} \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^T (T-\tau)^{-\alpha\gamma'} \left\| f(u(\tau)) \right\|_{\mu+2-4\gamma\gamma'}^2 d\tau \\ &\leq M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} t^{-2\alpha\gamma'} \int_0^T (T-\tau)^{-\alpha\gamma'} L_f(\|u(\tau)\|_\mu, 0)^2 \|u(\tau)\|_\mu^2 d\tau \end{aligned}$$

$$\leq M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} t^{-2\alpha\gamma'} \int_0^T (T-\tau)^{-\alpha\gamma'} l_f(\tau^{\alpha\gamma'})^{-2} L_f(\tau^{\alpha\gamma'}\|u(\tau)\|_\mu, 0)^2 \|u(\tau)\|^2 d\tau$$

$$\leq M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} t^{-2\alpha\gamma'} \rho^{*2} \int_0^T (T-\tau)^{-\alpha\gamma'} \tau^{-2\alpha\gamma'} l_f(\tau^{\alpha\gamma})^{-2} L_f(\tau^{\alpha\gamma'}\|u(\tau)\|_\mu, 0)^2 d\tau$$

$$\leq M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} t^{-2\alpha\gamma'} \rho^{*2} (L_f^{*2} + \zeta) \int_0^T (T-\tau)^{-\alpha\gamma'} \tau^{-2\alpha\gamma'} l_f(\tau^{\alpha\gamma'})^{-2} d\tau. \quad (23)$$

To deal with $I_3(t)$, we use Lemma 2.1. One can see that

$$\| S * f(u)(t) \|_\mu^2 \leq M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^t (t-\tau)^{-\alpha\gamma'} \| f(u(\tau)) \|_{\mu-2\gamma\gamma}^2 d\tau$$

$$\leq \lambda_1^{2\gamma\gamma'-2} M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^t (t-\tau)^{-\alpha\gamma'} \| f(u(\tau)) \|_{\mu+2-4\gamma\gamma'}^2 d\tau.$$

By using the same arguments as in the estimate for $I_2(t)$, one can have

$$I_3(t) \leq \lambda_1^{2\gamma\gamma'-2} M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \rho^{*2} (L_f^{*2} + \zeta) \int_0^t (t-\tau)^{-\alpha\gamma'} \tau^{-2\alpha\gamma'} \ell_f(\tau^{\alpha\gamma'})^{-2} d\tau. \quad (24)$$

From (22)-(24) we obtain

$$t^{2\alpha\gamma'} \| \Phi(u)(t) \|_\mu^2$$
$$\leq 3M_0 \| \xi \|_{\mu+2(1-\gamma\gamma')}^2 + 3(L_f^{*2} + \zeta)[M_0 M_1 \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \Lambda(T)$$
$$+ \lambda_1^{2\gamma\gamma'-2} M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} t^{2\alpha\gamma'} \Lambda(t)]\rho^{*2}$$
$$\leq 3M_0 \| \xi \|_{\mu+2(1-\gamma\gamma')}^2 + \frac{1}{2}\rho^{*2}.$$

Choose $\beta = \left(\sqrt{6M_0}\right)^{-1} \rho^*$, with $\| \xi \|_{\mu+2(1-\gamma\gamma')} \leq \beta$ we have

$$t^{2\alpha\gamma'} \| \Phi(u)(t) \|_\mu^2 \leq \rho^{*2} \text{ for all } t \in (0, T].$$

Therefore $\Phi(u) \in B_{\rho^*}$.

The second step is showing that Φ is a contraction mapping on B_{ρ^*}. For $u, v \in B_{\rho^*}$, one gets

$$\| \Phi(u)(t) - \Phi(v)(t) \|_\mu^2$$
$$\leq 2 \| P(t)(S * [f(u) - f(v)](T)) \|_\mu^2 + 2 \| S * [f(u) - f(v)](t) \|_\mu^2$$
$$\leq 2M_0 M_1 t^{-2\alpha\gamma'} \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^T (T-\tau)^{-\alpha\gamma'} \| f(u(\tau)) - f(v(\tau)) \|_{\mu+2-4\gamma\gamma'}^2 d\tau$$
$$+ 2\lambda_1^{2\gamma\gamma'-2} M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} \int_0^t (t-\tau)^{-\alpha\gamma'} \| f(u(\tau)) - f(v(\tau)) \|_{\mu+2-4\gamma\gamma'}^2 d\tau.$$

Notice that

$$\begin{aligned}\int_0^t & (t-\tau)^{-\alpha\gamma'} \| f(u(\tau)) - f(v(\tau)) \|_{\mu+2-4\gamma\gamma'}^2 \, d\tau \\ & \le \int_0^t (t-\tau)^{-\alpha\gamma'} L_f\big(\| u(\tau) \|_\mu, \| v(\tau) \|_\mu\big)^2 \| u(\tau) - v(\tau) \|_\mu^2 \, d\tau \\ & \le \int_0^t (t-\tau)^{-\alpha\gamma'} \tau^{-2\alpha\gamma'} \ell(\tau^{\alpha\gamma})^{-2} \\ & \qquad \times L_f\big(\tau^{\alpha\gamma'} \| u(\tau) \|_\mu, \tau^{\alpha\gamma'} \| v(\tau) \|_\mu\big)^2 \big[\tau^{2\alpha\gamma'} \| u(\tau) - v(\tau) \|_\mu^2\big] d\tau \\ & \le \big({L_f^*}^2 + \zeta\big) \Lambda(t) \| u - v \|_{\mu,\alpha\gamma'}^2.\end{aligned}$$

Therefore

$$\begin{aligned}t^{2\alpha\gamma'} \| \Phi(u)(t) - \Phi(v)(t) \|_\mu^2 & \le 2\big({L_f^*}^2 + \zeta\big)[M_0 M_1 t^{-2\alpha\gamma'} \frac{T^{1-\alpha\gamma'}}{1-\alpha\gamma'} \Lambda(T) \\ & \quad + \lambda_1^{2\gamma\gamma'-} \; M_1 \frac{t^{1-\alpha\gamma'}}{1-\alpha\gamma'} t^{2\alpha\gamma'} \Lambda(t)] \| u - v \|_{\mu,\alpha\gamma'}^2 \\ & \le \frac{1}{3} \| u - v \|_{\mu,\alpha\gamma}^2 \quad \text{for all } t \in (0, T],\end{aligned}$$

this implies that Φ is a contraction mapping on B_{ρ^*}. The proof is complete.

Example 3.3. We now give an example of a function f that satisfies the assumption **(F)** and the assumptions of Theorem 3.2. Let Ω be a subset of $\mathbb{R}^d$, $d \ge 2$. The parameters $\mu, \theta, \gamma, \gamma'$ are given in Theorem 3.2. Consider the function $f(u) = |u|^p$.

For $\theta = 4\gamma\gamma' - 2 - \mu > 0$, if we put $q = \frac{2d}{d+2\theta}$, $\hat{p} = \frac{2d}{d-2\mu}$, $\hat{q} = \frac{d}{\mu+\theta}$, then $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = \frac{1}{q}$. Applying the general Hölder inequality for $u \in L^{(p-1)\hat{q}}(\Omega), v \in L^{\hat{p}}(\Omega)$, we get

$$\begin{aligned}\| |u|^{p-1} v \|_{L^q} & \le \| |u|^{p-1} \|_{L^{\hat{q}}} \| v \|_{L^{\hat{p}}} \\ & = \| u \|_{L^{(p-1)\hat{q}}}^{p-1} \| v \|_{L^{\hat{p}}}.\end{aligned}$$

Assume that $(p-1)\hat{q} \le \hat{p}$, then it follows from [20, Lemma 3] that $\mathbb{H}^\mu \subset H_0^\mu(\Omega) \subset L^{\hat{p}}(\Omega) \subset L^{(p-1)\hat{q}}(\Omega)$. Hence $\| |u|^{p-1} v \|_{L^q} \le C_1 \| u \|_{\mathbb{H}^\mu}^{p-1} \| v \|_{\mathbb{H}^\mu}$, where C_1 is a positive constant which is not depend on u and v. Moreover, we also have $L^q(\Omega) \subset H^{-\theta} \subset \mathbb{H}^{-\theta}$ (from [20, Lemma 3]). Therefore there exists a constant C_2 such that $\| |u|^{p-1} v \|_{\mathbb{H}^{-\theta}} \le C_2 \| u \|_{\mathbb{H}^\mu}^{p-1} \| v \|_{\mathbb{H}^\mu}$. With $u, v \in C((0,T]; \mathbb{H}^\mu)$, one gets

$\| |u(t)|^p - |v(t)|^p \|_{\mathbb{H}^{-\theta}} \le C_3\big(\| u(t) \|_{\mathbb{H}^\mu}^{p-1} + \| v(t) \|_{\mathbb{H}^\mu}^{p-1}\big) \| u(t) - v(t) \|_{\mathbb{H}^\mu}$, where C_3 is a constant.

So the condition **(F)** holds for $L_f(\rho, \rho') = C_3(\rho^{p-1} + \rho'^{p-1})$ and $\ell_f(\lambda) = \lambda^{p-1}$.

4. Conclusion

In this paper, we study the mild solution of the final value problem for the reaction-subdiffusion equation. The main contributions of this paper include several new estimates related to the resolvent operator in Hilbert scales and sufficient conditions for the unique existence of mild solution. This research issue allows us to consider broader classes of nonlinear functions, which play important roles in physics. In subsequent works, we want to study the regularity with respect to the time variable of the solution, specifically Hölder regularity.

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