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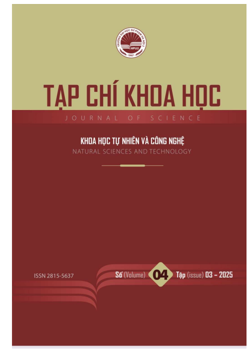

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A finiteness theorem for ends of weighted manifolds with a weighted Poincaré inequality

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Abstract

In this paper, we study complete weighted manifolds that satisfy a weighted Poincaré inequality, with the associated weight function assumed to be non-negative throughout the manifold. Our main focus is to study the geometric consequences of such an inequality on the global structure of the manifold, particularly at infinity. Specifically, we prove that such a manifold has only finitely many ϕ-nonparabolic ends, provided that the Bakry-Émery Ricci curvature is bounded from below outside a compact subset with respect to the weight function. This result generalizes several existing theorems in the theory of Riemannian geometry and offers valuable insight into the interplay between curvature conditions and the topology of ends.

Keywords: Weighted manifolds, weighted Poincaré inequality, finiteness theorems, ϕ-harmonic functions, ϕ-nonparabolic ends

1. Introduction

A weighted manifold is defined as a triple $\left(M, ds_M^2, e^{-\phi} d\mu\right)$, where $\left(M, ds_M^2\right)$ denotes a complete Riemannian manifold of dimension $n \geq 3$ and $e^{-\phi} d\mu$ is a weighted measure determined by a smooth potential function ϕ, with $d\mu$ being the standard Riemannian volume measure. Within this setting, the ϕ-Laplacian Δ_ϕ is given by $\Delta_\phi := \Delta \cdot - \langle \nabla\phi, \nabla \cdot \rangle$, which naturally generalizes the Laplace-Beltrami operator Δ to the framework of weighted manifolds. This operator reduces to the classical Laplacian

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exactly when the potential function ϕ is constant. The ϕ-Laplacian is self-adjoint with respect to the weighted measure $e^{-\phi}d\mu$. More precisely, for any smooth functions $\omega, v \in \mathcal{C}_0^\infty(M)$ and, one has

$$\int_M \omega\left(\Delta_\phi v\right)e^{-\phi}d\mu = -\int_M \langle \nabla v, \nabla \omega\rangle e^{-\phi}d\mu.$$

A smooth function ω defined on M is called ϕ-harmonic when it obeys the equation $\Delta_\phi \omega = 0$. The bottom spectrum of the weighted Laplacian Δ_ϕ can be characterized by

$$\lambda_1\left(\Delta_\phi\right) = \inf_{\omega \in \mathcal{C}_0^\infty(M)} \frac{\int_M |\nabla \omega|^2 e^{-\phi}d\mu}{\int_M \omega^2 e^{-\phi}d\mu}.$$

The Bakry-Émery Ricci curvature associated to the weighted manifold $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ was first introduced in [1] and is given by $\mathrm{Ric}_\phi = \mathrm{Ric} + \mathrm{Hess}\phi$, where Ric denotes the Ricci curvature of M and $\mathrm{Hess}\phi$ is the Hessian matrix of ϕ with respect to the metric tensor ds_M^2. This curvature is also related to the gradient Ricci soliton $\mathrm{Ric}_\phi = \lambda g$ for some constant λ, which has a fundamental role in the analysis of singularities of the Ricci flow. A gradient Ricci soliton is categorized as expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, respectively. Gradient Ricci solitons constitute a natural generalization of Einstein manifolds and, in recent years, have become a central focus of research in geometric analysis; see [2] for a comprehensive survey and additional references. A fundamental analytic tool in their study is the weighted Bochner-Weitzenböck formula, which plays a crucial role in deriving gradient estimates, rigidity results, and various comparison theorems for manifolds with density. Specifically, for any function $\omega \in \mathcal{C}^\infty(M)$, we have

$$\frac{1}{2}\Delta_\phi |\nabla \omega|^2 = \left|\mathrm{Hess}\ \omega\right|^2 + \left\langle \nabla \omega, \nabla \Delta_\phi \omega\right\rangle + \mathrm{Ric}_\phi(\nabla \omega, \nabla \omega).$$

Because $\mathrm{tr}\left(\mathrm{Hess}\,\omega\right) \neq \Delta_\phi \omega$, geometric comparison results cannot be derived in the same way as in the classical Ricci curvature setting on Riemannian manifolds. Despite this challenge, in [3], Wei and Wylie established several weighted mean curvature comparison theorems that extend the classical results, under the assumptions that the Bakry-Émery Ricci curvature is bounded from below and that either the potential function or its gradient is bounded. Subsequent to their contributions, a wide range of classical geometric and topological results pertaining to manifolds with Ricci curvature bounded from below have been extended to the broader framework of weighted manifolds. These generalizations, however, typically require additional conditions on the potential function ϕ. For further discussion on weighted manifolds, we refer the reader to [3]–[10] and the references therein.

A central focus of geometric analysis is to investigate the intricate relationship between the geometry and topology of manifolds, employing L^2 harmonic forms in tandem with harmonic functions. There have been interesting results in this direction, which have been expanded and generalized on weighted manifolds; see [4]–[6], [8]–[17] and the references therein. In [4], Dung and Sung investigated complete weighted manifolds that satisfied both the Bakry-Émery curvature lower bound and the following weighted Poincaré inequality:

Definition 1.1. Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be an n-dimensional complete weighted Riemannian manifold with $n \geq 3$. We say that M satisfies a weighted Poincaré inequality with a non-negative weight function $\rho(x)$ and a positive constant A if the following inequality holds for all compactly supported smooth functions $\phi \in \mathcal{C}_0^\infty(M)$:

$$A\int_M \rho(x)\phi^2(x)e^{-\phi(x)}d\mu \leq \int_M |\nabla\phi|^2(x)e^{-\phi(x)}d\mu \tag{1}$$

In this case, we say that M has property $(\mathcal{P}_{\rho,A})$. Moreover, the associated (ρ, A)-metric, defined by $ds_{\rho,A}^2 = A\rho ds_M^2$, is assumed to be complete.

Obviously, when $\rho(x) \equiv \dfrac{\lambda_1(\Delta_\phi)}{A}$ is a positive constant, M is a weighted manifold with positive spectrum of the weighted Laplacian. With respect to the metric $ds_{\rho,A}^2$, the (ρ, A)-distance between two points $x, y \in M$ is defined as $r_{\rho,A}(x,y) := \inf_\gamma l_{\rho,A}(\gamma)$ where the infimum is taken over all smooth curves γ joining x and y, and $l_{\rho,A}(\gamma)$ denotes the length of γ measured with $ds_{\rho,A}^2$. For a fixed point $p \in M$, we set $r_{\rho,A}(x) := r_{\rho,A}(p,x)$ to be the (ρ, A)-distance from p to x. For all $n \geq 3$ and $R > 0$, we define

$$F(R) = \exp\left(\frac{(n-1)(n-3)+2\sqrt{(n-2)}a}{(n-2)(n-1)+\sqrt{(n-2)}a}R\right),$$

and $S(R) = \sup_{B_{\rho,A}(R)} \sqrt{\rho}$ to be the maximum value of $\sqrt{\rho}$ over the geodesic ball $B_{\rho,A}(p,R)$ of radius R with respect to the metric $ds_{\rho,A}^2$ centered at a fixed point p. In [4], the authors proved the following rigidity result.

Theorem 1.2. [4, Theorem 1.3] Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be a complete weighted manifold of dimension $n \geq 3$. Assume for some nonzero weight function $\rho \geq 0$ and constant $a \in \left[0, \dfrac{n-1}{\sqrt{n-2}}\right)$ that M satisfies the property $\left(\mathcal{P}_{\rho,A}\right)$ with

$$A \geq \min\left\{n-1, \left(\sqrt{n-2} + \frac{a}{n-1}\right)^2\right\}.$$

For all $x \in M$, suppose that the following inequalities hold:

$$|\nabla\phi|(x) \leq a\rho^{\frac{1}{2}}(x) \text{ and } \operatorname{Ric}_\phi(x) \geq -(n-1)\rho(x).$$

If the weight function ρ satisfies the growth condition

$$\liminf_{R\to\infty} \frac{S(R)}{F(R)} = 0,$$

then either

M has only one ϕ-nonparabolic end, or

M is isometric to a warped product $M=\mathbb{R}\times N^{n-1}$ with metric

$$ds_M^2 = dt^2 + \eta^2(t)ds_N^2,$$

for some positive function $\eta(t)$, and some compact manifold N^{n-1}.

Remark 1.3. While Theorem 1.3 in [4] does not explicitly mention the condition $a\in\left[0,\frac{n-1}{\sqrt{n-2}}\right)$, this assumption is in fact essential for the application of Hölder's inequality in its proof (see page 625 of [4]).

By relaxing the condition of the Bakry-Émery curvature in Theorem 1.2 to be only satisfied outside a compact set of M, in this paper, we obtain a finiteness result for the ϕ-nonparabolic ends of the weighted manifold. This result may be viewed as a generalization of Theorem 1 in [18], extending it from Riemannian manifolds to smooth metric measure spaces.

Theorem 1.4. Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be a complete weighted manifold of dimension $n\geq 3$. Assume for some nonzero weight function $\rho\geq 0$ and constant $a\in\left[0,\frac{n-1}{\sqrt{n-2}}\right)$ that M satisfies propert$\left(\mathcal{P}_{\rho,A}\right)$ with

$$A\geq \min\left\{n-1,\left(\sqrt{n-2}+\frac{a}{n-1}\right)^2\right\}.$$

For all $x\in M$, suppose that the following inequalities hold:

$$|\nabla\phi|(x)\leq a\rho^{\frac{1}{2}}(x) \text{ and } \mathrm{Ric}_\phi^{M\setminus K}(x)\geq -(n-1)\rho(x)+\tilde{\varepsilon}$$

for some $\tilde{\varepsilon}>0$, compact set $K\subseteq M$. If the weight function ρ satisfies the growth estimate

$$\liminf_{R\to\infty}\frac{S(R)}{F(R)}=0,$$

then M has only finitely many ϕ-nonparabolic ends.

When $A=1$, the property $\left(\mathcal{P}_{\rho,A}\right)$ becomes the property $\left(\mathcal{P}_{\rho}\right)$. By using the same arguments as in the proof of Theorem 1.4, we can obtain the following result, which can be seen as a generalization of Theorem 1.4.

Theorem 1.5. Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be an n-dimensional $(n\geq 3)$ weighted manifold with property $\left(\mathcal{P}_{\rho}\right)$ for some non-zero weight function $\rho\geq 0$. For all $x\in M$, suppose that the following inequalities hold:

$$|\nabla\phi|(x)\leq a\tau^{\frac{1}{2}}(x) \text{ and } \mathrm{Ric}_\phi^{M\setminus K}(x)\geq -(n-1)\tau(x)+\tilde{\varepsilon}$$

for some $\tilde{\varepsilon}>0$, compact set $K\subseteq M$. Also assume that and the function $\tau(x)$ satisfies the following Poincare inequality

$$A\int_M \tau\varphi^2 e^{-\phi} d\mu \le \int_M |\nabla\varphi|^2 \, e^{-\phi} d\mu, \quad \forall \varphi \in C_0^\infty(M),$$

with

$$A \ge \min\left\{ n-1, \left(\sqrt{n-2} + \frac{a}{n-1} \right)^2 \right\}.$$

If ρ and τ satisfy the growth condition

$$\liminf_{R\to\infty} \frac{\mathcal{S}(R)}{F(R)} = 0$$

then M has only finitely many ϕ-nonparabolic ends, where $\mathcal{S}(R) = \sup_{B_\rho(R)} \left(\sqrt{\rho}, \sqrt{|\tau|} \right)$.

This section concludes with a brief overview of the paper's structure. Section 2 is devoted to preliminary results concerning ends and ϕ-harmonic functions, along with the relationship between them. In Section 3, we follow the methodology introduced in [18] to prove Theorem 1.4.

2. Ends and ϕ-harmonic functions

In this section, we review fundamental results on ends of manifolds, ϕ-harmonic functions, and related topics, which will be essential for proving our main theorem. The primary references for this section are [3], [4], [7], [9]–[11], [14]. Let $\left(M, ds_M^2, e^{-\phi} d\mu\right)$ be a complete weighted Riemannian manifold of dimension $n \ge 3$. Throughout this work, we denote by $B(o,R)$ the geodesic ball of radius $R > 0$ centered at a point $o \in M$, measured with respect to the original Riemannian metric ds_M^2:

$$B(o,R) := \{ x \in M \mid r(o,x) < R \},$$

where $r(o,x)$ denotes the Riemannian distance from o to x in the metric ds_M^2. Similarly, let $ds_{\rho,A}^2$ denote the conformally changed metric associated with a weight function ρ and parameter $A > 0$. We then define the corresponding geodesic ball $B_{\rho,A}(o,\kappa)$ by

$$B_{\rho,A}(o,R) := \{ x \in M \mid r_{\rho,A}(o,x) < R \},$$

where $r_{\rho,A}(o,x)$ denotes the geodesic distance from o to x with respect to the metric $ds_{\rho,A}^2$. For simplicity of notation, when the base point p is fixed and understood from the context, we shall write

$$B(R) := B(o,R), \qquad B_{\rho,A}(R) := B_{\rho,A}(o,R).$$

In addition, for any measurable subset $\Omega \subset M$, we denote by $V(\Omega)$ the standard Riemannian volume of Ω with respect to the volume element $d\mu$, and by $V_\phi(\Omega)$ the weighted volume (or ϕ-volume) Ω with respect to the weighted measure $e^{-\phi} d\mu$.

We begin by providing the definition of ends.

Definition 2.1. On $\left(M, ds_M^2, e^{-\phi}d\mu\right)$, a smooth function $\mathcal{G}_\phi(w,z)$ defined on $(M\times M)\setminus\{(w,w)\}$ is said to be ϕ-Green's function if it satisfies

$$\mathcal{G}_\phi(w,z)=\mathcal{G}_\phi(z,w) \text{ and } \Delta_{\phi,z}\mathcal{G}(w,z)=-\delta_{\phi,w}(z)$$

for all $x\neq y$, where $\delta_{\phi,w}(z)$ is defined by

$$\int_M \psi(z)\delta_{\phi,w}(z)e^{-\phi}d\mu=\psi(w)$$

for every compactly supported function $\psi\subset M$.

By applying arguments similar to those used in the proof of Theorem 1 in [19], one can show that every complete weighted manifold admits a ϕ-Green's function. However, while some of these spaces possess a positive ϕ-Green's function, others do not. This intriguing distinction has naturally divided the function theory of weighted manifolds into two separate classes.

Definition 2.2. A complete weighted Riemannian manifold $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ is said to be ϕ-nonparabolic if it admits a positive ϕ-Green's function. Otherwise, it is called ϕ-parabolic.

More generally, an end of M refers to an unbounded connected component of the complement of a smooth compact domain in M.

Definition 2.3. Let E be an end of the complete weighted manifold $\left(M, ds_M^2, e^{-\phi}d\mu\right)$. The end E is said to be ϕ nonparabolic if there exists a positive ϕ-Green's function on E satisfying Neumann boundary conditions on ∂E. Otherwise, E is called ϕ-parabolic.

If E is an end of M, we denote

$$E_{\rho,A}(R):=E\cap B_{\rho,A}(R) \text{ and } \partial E_{\rho,A}(R):=E\cap\partial B_{\rho,A}(R),$$

where $B_{\rho,A}(R)$ is the geodesic ball of radius R with respect to the conformal metric $ds_{\rho,A}^2$.

From the result of Seo-Yun in [20, Lemma 3.1], we know that the number of ϕ-nonparabolic ends of a weighted manifold $(M, ds_M^2, e^{-\phi}d\mu)$ is bounded above by the dimension of the space $\mathcal{K}^0(M)$ of bounded ϕ-harmonic functions with finite weighted Dirichlet integral, that is,

$$\sharp(\phi\text{-nonparabolic ends of } M)\leq \dim\mathcal{K}^0(M). \qquad (2)$$

The following lemma plays an important role in proving Theorem 1.4, which can be seen as a weighted version of Lemma 11 by Li in [21].

Lemma 2.4. [21, Lemma 11] Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be an n-dimensional $(n\geq 3)$ complete weighted manifold. Let $H^1\left(L_\phi^2(M)\right)$ be the space of L^2-integrable ϕ-harmonic 1-forms on M. If $\mathcal{H}$ is a finite dimensional subspace of $H^1\left(L_\phi^2(M)\right)$ defined over a set $\Omega\subseteq M$ then there exists $\omega_0\in\mathcal{H}$ such that

$$\dim\mathcal{H}\int_\Omega|\omega_0|^2e^{-\phi}dv\leq V_\phi(\Omega)\cdot\min\{n,\dim\mathcal{H}\}\cdot\sup_\Omega|\omega_0|^2.$$

Towards the end of this section, we demonstrate the following result, which constitutes a key step in the proof of the main theorem.

Lemma 2.5. Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be a complete weighted Riemannian manifold of dimension $n \geq 3$, and suppose that M satisfies the property $\left(\mathcal{P}_{\rho,A}\right)$ for some nontrivial weight function $\rho \geq 0$. Assume that the following pointwise conditions hold for all $x \in M$:

$$|\nabla\phi|(x) \leq a\rho^{\frac{1}{2}}(x) \text{ and } \mathrm{Ric}_{\phi}(x) \geq -(n-1)\rho(x),$$

where $a \geq 0$ is a constant. Then, for every positive ϕ-harmonic function ω on M, there exists a constant $C > 0$, depending solely on n, a and A, for which the gradient estimate

$$\left|\nabla\omega(x)\right| \leq CS(R+1)\,\omega(x) \tag{3}$$

holds for all $x \in B_{\rho,A}(R)$.

Proof of Lemma 2.5. Applying Theorem 2.1 in Wu [16], we conclude that there exist constants C_1, C_2 such that

$$\frac{|\nabla\omega|}{\omega}(x) \leq C_1 \sup_{B(x,R)} \sqrt{\rho} + \frac{C_2}{R}, \tag{4}$$

for all $x \in M$, where C_1, C_2 are constants depending only on n and a. For $t \in (0,\infty)$, we consider the following function

$$\mathcal{F}(t) = t - \left(\sup_{B(x,t)} \sqrt{A\rho}\right)^{-1}.$$

We observe that $\lim_{t\to 0}\mathcal{F}(t) = 0$ and $\lim_{t\to\infty}\mathcal{F}(t) = \infty$. Thus, by Mean value theorem, we can choose $R_0 > 0$ such that $\mathcal{F}(R_0) = 0$, that is

$$R_0 = \left(\sup_{B(x,R_0)} \sqrt{A\rho}\right)^{-1}.$$

For any point $y \in B(x, R_0)$, let γ be a minimizing geodesic (with respect to ds_M^2) joining x, y. Then, we have

$$r_{\rho,A}(x,y) = \int_\gamma \sqrt{A\rho(\gamma(t))}dt \leq \sup_{B(x,R_0)} \sqrt{A\rho(y)}R_0 \leq 1.$$

This shows that $B(x, R_0) \subset B_{\rho,A}(x,1)$. For any $x \in B_{\rho,A}(R)$, we see that

$$B(x, R_0) \subset B_{\rho,A}(x,1) \subset B_{\rho,A}(R+1).$$

Now, we choose $R = R_0$ in (4) and conclude that

$$\frac{|\nabla\omega|}{\omega}(x) \leq \widetilde{C}\left(\sup_{B(x,R_0)} \sqrt{\rho} + \frac{1}{R_0}\right) = \widetilde{C}\left(1+\sqrt{A}\right)\sup_{B(x,R_0)} \sqrt{\rho} \leq \widetilde{C}\left(1+\sqrt{A}\right)\sup_{B_{\rho,A}(R+1)} \sqrt{\rho} = CS(R+1),$$

where $\widetilde{C} = \max\{C_1, C_2\}$. The proof of Lemma 2.5 is complete.

The proof of Theorem 1.5 follows essentially the same approach as that of Theorem 1.4, with only a minor difference stemming from Lemma 2.5. To conclude this section, we provide an alternative formulation of Lemma 2.5, that is tailored to the assumptions of Theorem 1.5.

Lemma 2.6. Let $\left(M, ds_M^2, e^{-\phi}d\mu\right)$ be an n-dimensional $\left(n\geq 3\right)$ weighted manifold with property $\left(\mathcal{P}_\rho\right)$ for some non-zero weight function $\rho\geq 0$. For all $x\in M$, suppose that the following inequalities hold:

$$|\nabla\phi|(x)\leq a\tau^{\frac{1}{2}}(x) \text{ and } \mathrm{Ric}_\phi^{M\setminus K}(x)\geq -(n-1)\tau(x)+\tilde{\varepsilon}$$

for some $\tilde{\varepsilon}>0$, compact set $K\subseteq M$. Also assume that and the function $\tau(x)$ satisfies the following Poincare inequality

$$A\int_M \tau\varphi^2 e^{-\phi}d\mu\leq \int_M |\nabla\varphi|^2\, e^{-\phi}d\mu,\quad \forall\varphi\in C_0^\infty(M),$$

with

$$A\geq \min\left\{n-1, \left(\sqrt{n-2}+\frac{a}{n-1}\right)^2\right\}.$$

Then, for every positive ϕ-harmonic function ω on M, there exists a constant $C>0$, depending solely on n and a, for which the gradient estimate

$$\left|\nabla\omega(x)\right|\leq C\mathcal{S}(R+1)\omega(x)$$

holds for all $x\in B_\rho(R)$, where $\mathcal{S}(R)=\sup\limits_{B_p(R)}\left(\sqrt{\rho},\sqrt{|\tau|}\right)$.

Proof of Lemma 2.6. As in the proof of Lemma 2.4, there exists a constant C, depending only on n and a, such that

$$\frac{|\nabla\omega|}{\omega}(x)\leq C\left(\sup_{B(x,R)}\sqrt{\tau}+\frac{1}{R}\right).$$

For all $x\in M$, set

$$\bar{\rho}\left(x\right)=\frac{1}{2}\rho(x)+\frac{1}{2}|\tau(x)|.$$

Then, we have $\sqrt{|\tau|}\leq\sqrt{2\bar{\rho}}$, $\sup\limits_{B_\rho(R)}\sqrt{\bar{\rho}}\leq \sup\limits_{B_\rho(R)}\left(\sqrt{\rho},\sqrt{|\tau|}\right)=\mathcal{S}(R)$, and

$$\frac{|\nabla\omega|}{\omega}(x)\leq C\left(\sup_{B(x,R)}\sqrt{\bar{\rho}}+\frac{1}{R}\right),$$

for all $x\in M$. For $t\in\left(0,\infty\right)$, we consider the following function

$$\widetilde{\mathcal{F}}(t)=\sqrt{2}t-\left(\sup_{B(x,t)}\sqrt{\bar{\rho}}\right)^{-1}.$$

We see that $\lim\limits_{t\to 0}\widetilde{\mathcal{F}}(t)=0$ and $\lim\limits_{t\to\infty}\widetilde{\mathcal{F}}(t)=\infty$. Thus, by Mean value theorem, we can choose $R_0>0$ such that $\widetilde{\mathcal{F}}(R_0)=0$, that is

$$\sqrt{2}R_0 = \left(\sup_{B(x,R_0)} \sqrt{\overline{\rho}} \right)^{-1}.$$

For any point $y \in B(x, R_0)$, let γ be a minimizing geodesic (with respect to ds_M^2) joining x, y. Then, we have

$$r_\rho(x,y) = \int_\gamma \sqrt{\rho(\gamma(t))}dt \le \int_\gamma \sqrt{2}\sqrt{\overline{\rho}(\gamma(t))}dt \le \sqrt{2}R_0 \sup_{B(x,R_0)} \sqrt{\overline{\rho}} = 1.$$

This shows that $B(x,R_0) \subset B_\rho(x,1)$. For any $x \in B_\rho(R)$, we obtain

$$B(x,R_0) \subset B_\rho(x,1) \subset B_\rho(R+1).$$

Now, we choose $R = R_0$ in the inequality $\frac{|\nabla\omega|}{\omega}(x) \le C\left(\sup_{B(x,R)} \sqrt{\rho} + \frac{1}{R} \right)$ and conclude that

$$\frac{|\nabla\omega|}{\omega}(x) \le C\left(\sup_{B(x,R_0)} \sqrt{\overline{\rho}} + \frac{1}{R_0} \right) = C\left(1+\sqrt{2}\right) \sup_{B(x,R_0)} \sqrt{\overline{\rho}} \le C\left(1+\sqrt{2}\right) \sup_{B_\rho(R+1)} \sqrt{\overline{\rho}} = C\left(1+\sqrt{2}\right)\mathcal{S}(R+1).$$

for all $x \in M$. The proof of Lemma 2.6 is complete.

3. Proof of Theorem 1.4

With the necessary preparations in place, we proceed to prove Theorem 1.4 by following the approach introduced by Lam in [18].

Proof of Theorem 1.4. In view of (2), to prove Theorem 1.4, it is sufficient to prove that $\dim \mathcal{K}^0(M) < \infty$. Assume henceforth that M admits at least two ϕ-nonparabolic ends. Following the construction in Subsection 2.1, we obtain a nonconstant bounded ϕ-harmonic function $\omega \in \mathcal{K}^0(M)$ with finite Dirichlet weighted integral, such that $\inf \omega = 0$ and $\sup \omega = 1$. Moreover, the infimum is attained at infinity of a ϕ-nonparabolic end E_1, while the supremum is attained at infinity of the other ϕ nonparabolic end $M \setminus E_1$.

If $A \ge n-1$, then, by arguments analogous to those employed in the initial part of the proof of Theorem 3.1 in [4], we deduce that the manifold M is isometric to the Riemannian product of a line and a compact manifold. Consequently, the proof of Theorem 1.4 is completed in this case.

We next turn to the case $A \ge \left(\sqrt{n-2} + \frac{a}{n-1} \right)^2$. Using Lemma 2.7 in [5], we get

$$\Delta_\phi |\nabla\omega| \ge -\left(n-1+\frac{a}{\sqrt{n-2}} \right)(\rho-\varepsilon)|\nabla\omega| + (1-\alpha)\frac{\left|\nabla|\nabla\omega|\right|^2}{|\nabla\omega|},$$

in $M \setminus K$, where $\alpha = \frac{n-2}{n-1} + \frac{\sqrt{n-2}\,a}{(n-1)^2}$ and $\varepsilon = \frac{1}{n-1}\tilde{\varepsilon}$. From this, we deduce that

$$\Delta_\phi h \ge -A(\rho-\varepsilon)h \tag{5}$$

in $M \setminus K$, where $h = |\nabla \omega|^{\alpha}$. Let $\phi \in \mathcal{C}_c^{\infty}(M \setminus K)$ be a non-negative smooth function with compact support in $M \setminus K$. Then by the property $(\mathcal{P}_{\rho,A})$, we have

$$A\int_M \rho\phi^2 h^2 e^{-\phi} d\mu \le \int_M |\nabla(\phi h)|^2 e^{-\phi} d\mu$$
$$= \int_M h^2 |\nabla \phi|^2 e^{-\phi} d\mu + 2\int_M h\phi \langle \nabla h, \nabla \phi \rangle e^{-\phi} d\mu + \int_M \phi^2 |\nabla h|^2 e^{-\phi} d\mu.$$

The Stokes' theorem implies that

$$2\int_M h\phi \langle \nabla h, \nabla \phi \rangle e^{-\phi} d\mu = \frac{1}{2}\int_M \left\langle \nabla h^2, \nabla \phi^2 \right\rangle e^{-\phi} d\mu$$
$$= -\int_M \phi^2 h \Delta_\phi h e^{-\phi} d\mu - \int_M \phi^2 |\nabla h|^2 e^{-\phi} d\mu.$$

From the above results, we deduce that

$$A\int_M \rho\phi^2 h^2 e^{-\phi} d\mu \le \int_M h^2 |\nabla \phi|^2 e^{-\phi} d\mu - \int_M \phi^2 h \Delta_\phi h e^{-\phi} d\mu.$$

This and (5) entail that

$$A\varepsilon \int_M \phi^2 h^2 e^{-\phi} d\mu \le \int_M h^2 |\nabla \phi|^2 e^{-\phi} d\mu \tag{6}$$

for any $\phi \in \mathcal{C}_c^{\infty}(M \setminus K)$. Since K is compact, we may choose $R_0 > 0$ such that

$$K \subseteq \bigcup_{x \in K} B_{\rho,A}(x,1) \subseteq B(R_0 - 1).$$

Let $R > 0$ be such that $B(R_0) \subseteq B_{\rho,A}(R-1)$. Then, from (6) we have

$$A\varepsilon \int_{B_{\rho,A}(R) \setminus B(R_0-1)} \phi^2 h^2 e^{-\phi} d\mu \le \int_{B_{\rho,A}(R) \setminus B(R_0-1)} h^2 |\nabla \phi|^2 e^{-\phi} d\mu,$$

for any $\phi \in \mathcal{C}_c^{\infty}\left(B_{\rho,A}(R) \setminus B(R_0 - 1)\right)$. We next choose $\phi = \chi\psi$ to be the product of two compactly supported functions, where

$$\chi(x) = \begin{cases} 0 & \text{on } \mathfrak{Y}(0, \delta\overline{\varepsilon}) \cup \mathfrak{Y}(1-\delta\overline{\varepsilon}, 1), \\ \dfrac{\ln(\delta\overline{\varepsilon}) - \ln u}{\ln \delta} & \text{on } \mathfrak{Y}(\delta\overline{\varepsilon}, \overline{\varepsilon}) \cap (M \setminus E_1), \\ \dfrac{\ln(\delta\overline{\varepsilon}) - \ln(1-u)}{\ln \delta} & \text{on } \mathfrak{Y}(1-\overline{\varepsilon}, 1-\delta\overline{\varepsilon}) \cap E_1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\psi(x) = \begin{cases} 0 & \text{on } B(R_0 - 1), \\ 1 & \text{on } B_{\rho,A}(R-1) \setminus B(R_0), \\ R - r_{\rho,A} & \text{on } B_{\rho,A}(R) \setminus B_{\rho,A}(R-1), \\ 0 & \text{on } M \setminus B_{\rho,A}(R), \end{cases}$$

for some $0 < \delta < 1$ and $0 < \overline{\varepsilon} < \dfrac{1}{2}$ to be determined later, where $\mathfrak{Y}(c,d) = \{x \in M : c < \omega(x) < d\}$, and $\mathfrak{y}(\tau) = \{x \in M : \omega(x) = \tau\}$. From this and the Cauchy-Schwarz inequality, we see that

$$\int_{B_{\rho,A}(R)\backslash B(R_0-1)} h^2 |\nabla\phi|^2 e^{-\phi} d\mu$$
$$\leq 2\int_{B_{\rho,A}(R)\backslash B(R_0-1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu + 2\int_{B_{\rho,A}(R)\backslash B(R_0-1)} h^2\psi^2 |\nabla\chi|^2 e^{-\phi} d\mu. \tag{7}$$

By the definition of the function ψ , we get

$$\int_{B_{\rho,A}(R)\backslash B(R_0-1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu$$
$$= \int_{(B_{\rho,A}(R)\backslash B_{\rho,A}(R-1))\cap E_1} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu + \int_{(B_{\rho,A}(R)\backslash B_{\rho,A}(R-1))\cap(M\backslash E_1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu \tag{8}$$
$$+ \int_{B(R_0)\backslash B(R_0-1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu.$$

In the following C denotes a constant depending only on n, a and, A whose value may change from line to line. Denote by

$$\frac{1}{p} := \alpha = \frac{n-2}{n-1} + \frac{\sqrt{n-2}}{(n-1)^2} a, \quad \frac{1}{q} := 1 - \frac{1}{p} = \frac{1}{n-1} - \frac{\sqrt{n-2}}{(n-1)^2} a.$$

Since $a \in \left[0, \frac{n-1}{\sqrt{n-2}}\right)$, we get $p, q \in (0,1)$. Employing Hölder's inequality together with the fact that $0 \leq \chi \leq 1$, the first term on the right-hand side of (8) admits the following estimate:

$$\int_{(B_{\rho,A}(R)\backslash B_{\rho,A}(R-1))\cap E_1} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu \leq \int_\Omega A\rho |\nabla u|^{\frac{2}{p}} e^{-\phi} d\mu$$
$$\leq A\left(\int_\Omega |\nabla u|^2 e^{-\phi} d\mu\right)^{\frac{1}{p}} \left(\int_\Omega \rho^q e^{-\phi} d\mu\right)^{\frac{1}{q}}, \tag{9}$$

where $\Omega = \left(B_{\rho,A}(R) \setminus B_{\rho,A}(R-1)\right) \cap E_1 \cap (\mathfrak{L}(\delta\bar{\varepsilon}, 1-\delta\bar{\varepsilon}))$. From Lemma 2.6 in [5], we have

$$\left(\int_\Omega |\nabla u|^2 e^{-\phi} d\mu\right)^{\frac{1}{p}} \leq C \exp\left(-\frac{2R}{p}\right). \tag{10}$$

By Lemma 2.6 in [5] and note that $S(R) = \sup_{B_{\rho,A}(R)} \sqrt{\rho}$, we get

$$\int_\Omega \rho^q e^{-\phi} d\mu \leq [S(R)]^{2(q-1)} \int_\Omega \rho e^{-\phi} d\mu$$
$$\leq [S(R)]^{2(q-1)} (\delta\bar{\varepsilon})^{-2} \int_\Omega \rho u^2 e^{-\phi} d\mu \leq C[S(R)]^{2(q-1)} (\delta\bar{\varepsilon})^{-2} \exp(-2R).$$

This implies that

$$\left(\int_\Omega \rho^q e^{-\phi} d\mu\right)^{\frac{1}{q}} \leq C[S(R)]^{\frac{2(q-1)}{q}} (\delta\bar{\varepsilon})^{-\frac{2}{q}} \exp\left(\frac{-2R}{q}\right). \tag{11}$$

Substituting (10) and (11) in (9), we deduce that

$$\int_{(B_{\rho,A}(R)\backslash B_{\rho,A}(R-1))\cap E_1} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu \leq C[S(R)]^{\frac{2(q-1)}{q}} (\delta\bar{\varepsilon})^{-\frac{2}{q}} \exp(-2R). \tag{12}$$

Following similar arguments as in the proof of the estimate (12), we also obtain

$$\int_{(B_{\rho,A}(R)\setminus B_{\rho,A}(R-1))\cap(M\setminus E_1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu \le C[S(R)]^{\frac{2(q-1)}{q}} (\delta\overline{\varepsilon})^{-\frac{2}{q}} \exp(-2R).$$

Plugging this and (12) into (8), we conclude that

$$\begin{aligned}&\int_{B_{\rho,A}(R)\setminus B(R_0-1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu \\ &\le C[S(R)]^{\frac{2(q-1)}{q}} (\delta\overline{\varepsilon})^{-\frac{2}{q}} \exp(-2R) + \int_{B(R_0)\setminus B(R_0-1)} h^2\chi^2 |\nabla\psi|^2 e^{-\phi} d\mu.\end{aligned} \tag{13}$$

On the other hand, since $0 \le \psi \le 1$, we obtain

$$\begin{aligned}\int_{B_{\rho,A}(R)\setminus B(R_0-1)} h^2\psi^2 |\nabla\chi|^2 e^{-\phi} d\mu &\le \int_{(B_{\rho,A}(R)\setminus B(R_0-1))\cap E_1} |\nabla u|^{\frac{2}{p}} |\nabla\chi|^2 e^{-\phi} d\mu \\ &+ \int_{(B_{\rho,A}(R)\setminus B(R_0-1))\setminus(M\setminus E_1)} |\nabla u|^{\frac{2}{p}} |\nabla\chi|^2 e^{-\phi} d\mu.\end{aligned} \tag{14}$$

Using the estimate (3) in Lemma 2.5, we have

$$\begin{aligned}\int_{(B_{\rho,A}(R)\setminus B(R_0-1))\cap(M\setminus E_1)} |\nabla u|^{\frac{2}{p}} |\nabla\chi|^2 e^{-\phi} d\mu &= (\ln\delta)^{-2} \int_{\widehat{\Omega}} u^{-2} |\nabla u|^{2+\frac{2}{p}} e^{-\phi} d\mu \\ &\le C(\ln\delta)^{-2} [S(R+1)]^{\frac{2}{p}} \int_{\widehat{\Omega}} |\nabla u|^2 u^{-2+\frac{2}{p}} e^{-\phi} d\mu,\end{aligned} \tag{15}$$

where $\widehat{\Omega} = \left(B_{\rho,A}(R) \setminus B\left(R_0-1\right)\right) \cap \left(M \setminus E_1\right) \cap \mathfrak{L}(\delta\overline{\varepsilon}, \overline{\varepsilon})$. The co-area formula and Lemma 2.8 in [5] entail that

$$\int_{\widehat{\Omega}} |\nabla u|^2 u^{-2+\frac{2}{p}} e^{-\phi} d\mu \le \int_{\delta\overline{\varepsilon}}^{\overline{\varepsilon}} t^{-2+\frac{2}{p}} \int_{\mathfrak{y}(t)\cap(M\setminus E_1)\cap B_{\rho,A}(R)} |\nabla u| \, dA dt \le \int_{\mathfrak{y}(b)} |\nabla u| \, dA \int_{\delta\overline{\varepsilon}}^{\overline{\varepsilon}} t^{-2+\frac{2}{p}} dt,$$

for any level set b. This and (15) imply that

$$\int_{(B_{\rho,A}(R)\setminus B(R_0-1))\cap(M\setminus E_1)} |\nabla u|^{\frac{2}{p}} |\nabla\chi|^2 e^{-\phi} d\mu \le C(\ln\delta)^{-2} [S(R+1)]^{\frac{2}{p}} \left(1-\delta^{\frac{2-p}{p}}\right) \overline{\varepsilon}^{\frac{2-p}{p}}. \tag{16}$$

Applying the same estimate as in the proof of Lemma 2.5 to the function $1-u$, we also have

$$|\nabla u(x)| \le CS(R+1)\left(1-u(x)\right) \tag{17}$$

for all $x \in B_{\rho,A}(R)$. Then, by replacing the function u with $1-u$ and using (3) in place of (17), we obtain the following estimate by applying the same reasoning as in the proof of inequality (16)

$$\int_{(B_{\rho,A}(R)\setminus B(R_0-1))\cap E_1} |\nabla u|^{\frac{2}{p}} |\nabla\chi|^2 e^{-\phi} d\mu \le C(\ln\delta)^{-2} [S(R+1)]^{\frac{2}{p}} \left(1-\delta^{\frac{2-p}{p}}\right) \overline{\varepsilon}^{\frac{2-p}{p}}.$$

From this, (14) and (16), we get

$$\int_{B_{\rho,A}(R)\setminus B(R_0-1)} h^2\psi^2 |\nabla\chi|^2 e^{-\phi} d\mu \le C(\ln\delta)^{-2} [S(R+1)]^{\frac{2}{p}} \left(1-\delta^{\frac{2-p}{p}}\right) \overline{\varepsilon}^{\frac{2-p}{p}}.$$

Hence together with (6), (7) and (13), we conclude that

$$A\varepsilon\int_{B_{\rho,A}(R)\backslash B(R_0-1)}\phi^2h^2e^{-\phi}d\mu\le\int_{B(R_0)\backslash B(R_0-1)}h^2\chi^2|\nabla\psi|^2e^{-\phi}d\mu$$
$$+C[S(R)]^{\frac{2(q-1)}{q}}(\delta\overline{\varepsilon})^{-\frac{2}{q}}\exp(-2R)+C(\ln\delta)^{-2}[S(R+1)]^{\frac{2}{p}}\left(1-\delta^{\frac{2-p}{p}}\right)\overline{\varepsilon}^{\frac{2-p}{p}}.$$

If we choose $\delta=\frac{1}{2}$ and $\overline{\varepsilon}=\exp(-2R)$ then the above inequality implies that

$$A\varepsilon\int_{B_{\rho,A}(R)\backslash B(R_0-1)}\phi^2h^2e^{-\phi}d\mu$$
$$\le C[S(R+1)]^{2\left(\frac{n-2}{n-1}+\frac{\sqrt{n-2}}{(n-1)^2}a\right)}\exp\left(-2R\left(\frac{n-3}{n-1}+\frac{2\sqrt{n-2}}{(n-1)^2}a\right)\right)+\int_{B(R_0)\backslash B(R_0-1)}h^2|\nabla\psi|^2e^{-\phi}d\mu \tag{18}$$

Given that $\lim_{R\to\infty}\frac{S(R)}{F(R)}=0,$ it follows that the first term on the right-hand side of the preceding inequality vanishes in the limit as $R\to\infty.$ Thus, letting $R\to\infty$ in (18), we obtain

$$A\varepsilon\int_{M\backslash B(R_0)}h^2e^{-\phi}d\mu\le C\int_{B(R_0)\backslash B(R_0-1)}h^2e^{-\phi}d\mu.$$

This deduces that

$$\int_{B(2R_0)}h^2e^{-\phi}d\mu\le C\int_{B(R_0)}h^2e^{-\phi}d\mu, \tag{19}$$

where $C=C(\varepsilon,n,a,A).$ From (19), we conclude that $\Delta_\phi h\ge-\beta h$ on $B(p,2R_0)$, where

$$\beta=\frac{A}{n-1}\inf_{B(p,2R_0)}\mathrm{Ric}_\phi.$$

Then by the mean value inequality of Wu [10, Theorem 5.2], we find that

$$h^2(x)\le C_1\int_{B(x,R_0)}h^2e^{-\phi}d\mu\le C_1\int_{B(p,2R_0)}h^2e^{-\phi}d\mu,$$

where $C_1=C_1(n,\beta,\mu)$ and $\mu=\inf_{x\in B(p,R_0)}V_\phi(B(x,R_0))$. This and (19) lead to

$$\sup_{B(p,R_0)}h^2\le C_2\int_{B(p,R_0)}h^2e^{-\phi}d\mu,$$

where $C_2=C_2(\varepsilon,\mu,\beta,n,a,A).$ Using the Hölder's inequality, we get

$$\int_{B(R_0)}h^2e^{-\phi}d\mu\le\left(\int_{B(R_0)}|\nabla\omega|^2e^{-\phi}d\mu\right)^{\alpha}\left[V_\phi(B(R_0))\right]^{1-\alpha}.$$

Consequently,

$$\sup_{B(R_0)}|\nabla\omega|^2\le C_3\int_{B(R_0)}|\nabla\omega|^2e^{-\phi}d\mu, \tag{20}$$

where $C_3=C_3(\varepsilon,\mu,\beta,n,a,A,R_0)$ is a constant independent of $u\in\mathcal{K}^0(M)$. We are now in a position to demonstrate that $\mathcal{K}^0(M)$ is finite dimensional. To this end, consider the space of 1-forms

$$\mathcal{K}=\left\{d\omega:\omega\in\mathcal{K}^0(M)\right\},$$

equipped with the bilinear form defined by $\int_{B(R_0)} \langle \nabla\omega, \nabla h\rangle e^{-\phi} d\mu$. Observe that if

$$\int_{B(R_0)} |\nabla\omega|^2 \, e^{-\phi} d\mu = 0$$

for some $\omega \in \mathcal{K}^0(M)$, then the unique continuation property implies that ω must be constant. Consequently, the bilinear form defines an inner product on $\mathcal{K}$. According to Lemma 2.4, there exists $\omega_0 \in \mathcal{K}^0(M)$ such that

$$\dim\mathcal{K}\int_{B(R_0)} |d\omega_0|^2 e^{-\phi} d\mu \le nV_\phi\left(B\left(R_0\right)\right) \sup_{B(R_0)} |d\omega_0|^2 .$$

From this and (20), one finds that

$$\dim\mathcal{K}^0(M) = \dim\mathcal{K} + 1 \le C_4$$

for some fixed constant $C_4 = C_4\left(C_3, V_\phi\left(B\left(R_0\right)\right)\right)$. The proof is complete.

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