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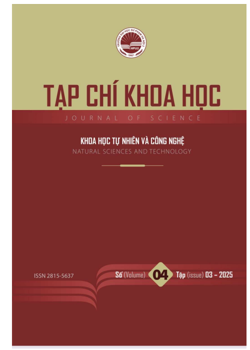

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The sequential Cohen-Macaulayness of idealizations

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. The *idealization* $R \ltimes M$, introduced by M. Nagata, has become a useful construction in commutative algebra. Recent work has characterized the approximate Cohen–Macaulayness of such idealizations via the length function associated with a good system of parameters. Motivated by these developments, we investigate via the sequential Cohen–Macaulayness of the idealization $R \ltimes M$. We provide a characterization in terms of the length function with respect to a good system of parameters of the form $(x_1, 0), \ldots, (x_r, 0)$, where $r = \dim R$. Furthermore, we provide equivalent conditions for $R \ltimes M$ to be sequentially Cohen–Macaulay, expressed in terms of the length functions of both R and M, and their respective dimension filtrations.

Keywords: Idealization, Dimension filtration, Good systems of parameters, Sequentially Cohen-Macaulay

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. On additive group $R \oplus M$, we define the multiplication as follows:

$$(r_1, m_1).(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$$

for all $(r_1, m_1), (r_2, m_2) \in R \oplus M$. Since R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, it follows that $R \oplus M$ is a Noetherian local ring with maximal ideal $\mathfrak{m} \times M$, and its dimension equals $\dim R$. Denoted by $R \ltimes M$ and referred to as the *idealization* of M over R, this local ring was introduced by M. Nagata [1] and has found important applications in commutative algebra. The study of the properties

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of idealization and its various applications has garnered significant attention from mathematicians (see [2]-[8]). Notably, the Gorensteinness of idealizations was studied by I. Reiten in [5]. Building on this, S. Goto et al. in [6] investigated the conditions under which the idealization $R \ltimes M$ qualifies as an almost Gorenstein local ring. Recently, P.H. Nam, D. V. Kien, and P.V. Loc [9] have characterized the approximate Cohen-Macaulayness of an idealization $R \ltimes M$ via the length function relative to a good system of parameters of the form $(x_1, 0), \ldots, (x_r, 0)$ (such a good system of parameters exists, see [9, Corollary 2.4], where $r = \dim R$.

Motivated by the work of P.H. Nam et al. (see [10]–[15]), we characterize the sequential Cohen-Macaulayness of an idealization in terms of the length function with respect to a good system of parameters of the form $(x_1, 0), \ldots, (x_r, 0)$.

To be more specific, let $\dim R = r$ and $\dim M = d$. Note that $\dim(R \ltimes M) = r \geq d$. If $r = 1$, then $R \ltimes M$ is a sequentially Cohen-Macaulay local ring. Therefore, we restrict our attention to the case $r \geq 2$. The following theorem constitutes the main result of this paper.

Theorem 1.1. *Suppose that $r \geq 2$. Put $A = R \ltimes M$. The following assertions are equivalent.*

(a) A is sequentially Cohen-Macaulay.

(b) There exists a good system of parameters $(x_1, 0), \ldots, (x_r, 0)$ of A such that

$$\ell\left(A / (u_1^{n_1}, \ldots, u_r^{n_r})\right) = \sum_{i=0}^{r} n_1 \ldots n_i e(u_1, \ldots, u_i; A_i)$$

for all $n_1, \ldots, n_r \geq 1$, where $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r$ is the dimension filtration of A and $u_1 = (x_1, 0), \ldots, u_r = (x_r, 0)$.

(c) One of the following conditions holds.

(i) $d = r$ and there exists a good system of parameters $x_1, \ldots, x_r$ of both R and M such that

$$\ell(R / (x_1^{n_1}, \ldots, x_r^{n_r})) = \sum_{i=0}^{r} n_1 \ldots n_i e(x_1, \ldots, x_i; R_i)$$

and

$$\ell(M / (x_1^{n_1}, \ldots, x_r^{n_r})M) = \sum_{i=0}^{r} n_1 \ldots n_i e(x_1, \ldots, x_i; M_i)$$

for all $n_1, \ldots, n_r \geq 1$, where $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_r$ and $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r$ are the dimension filtrations of R and M, respectively.

(ii) $d < r$ and there exists a good system of parameters $x_1, \ldots, x_r$ of R such that $x_1, \ldots, x_d$ is a good system of parameters of M and $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R M$ such that

$$\ell(R / (x_1^{n_1}, \ldots, x_r^{n_r})) = \sum_{i=0}^{r} n_1 \ldots n_i e(x_1, \ldots, x_i; R_i)$$

and

$$\ell(M / (x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} n_1 \ldots n_i e(x_1, \ldots, x_i; M_i)$$

for all $n_1,\ldots,n_r \ge 1$ *, where* $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_r$ *and* $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d$ *are the dimension filtrations of* R *and* M *, respectively.*

In the next section, we recall some properties of good system of parameters of the idealization $R \ltimes M$. In Section 3, we present the proof of Theorem 1.1.

2. Preliminaries

An interesting extension of Cohen-Macaulay modules is sequentially Cohen-Macaulay modules due to R.P. Stanley [16] for graded rings and P. Schenzel [17] for local rings. The study of sequential Cohen-Macaulayness has taken many different directions, see [13], [14], [18]–[22]. There is a close connection between this concept and the notion of dimension filtration introduced by P. Schenzel [17].

Let $0 \le i < d$ be an integer. Define M_i to be the largest submodule of M satisfying $\dim M_i \le i$. Since M is a Noetherian R -module, each submodule M_i is well-defined. Furthermore, these submodules form an ascending chain

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M,$$

where $M_{i-1} \subseteq M_i$ for all $i \in \{1,\ldots,d\}$. This ascending filtration of M is referred to as the *dimension filtration* of M (see [17, Definition 2.1]).

Definition 2.1 (see [17]). Let $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ be the dimension filtration of M . We say that M is a *sequentially Cohen-Macaulay module* if $M_i / M_{i-1} = 0$ or M_i / M_{i-1} is a Cohen-Macaulay module of dimension i for all $0 < i \le d$.

From now on, we denote $A = R \ltimes M$ as the idealization of M over R . Note that $\dim A = r$.
Then, we have the following lemma (see [9, Lemma 2.2]).

Lemma 2.2. *Let* $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ *and* $R_0 \subseteq R_1 \subseteq \ldots \subseteq R_r = R$ *be the dimension filtrations of* M *and* R *, respectively.*

(a) Let $d = r$*. For* $i = 0,\ldots,r$*, we put* $A_i = R_i \times M_i$*. Then, we have*

$$A_0 \subseteq A_1 \subseteq \ldots \subseteq A_r = A$$

is the dimension filtration of A *.*

(b) Let $d < r$*. For* $i = 0,\ldots,d$*, we put* $A_i = R_i \times M_i$ *and* $A_j = R_j \times M$ *for* $j = d+1,\ldots,r$*. Then, we have* $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_r = A$ *is the dimension filtration of* A *.*

Introduced by N.T. Cuong and D.T. Cuong in [18], the concept of a good system of parameters is a key tool in analyzing the sequential Cohen-Macaulay property of modules (see [19]).

Definition 2.3 (see [18, Lemma 2.2]). Let $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ be the dimension filtration and $\underline{x} = x_1,\ldots,x_d$ a system of parameters of M . $\underline{x}$ is called a *good system of parameters* of M if $M_i \cap (x_{i+1},\ldots,x_d)M = 0$ for $i \in 0,\ldots,d-1$.

Following [18, Lemma 2.5], M always admits a good system of parameters. By [9, Proposition 2.3], we can construct a good system of parameters for the idealization A .

Proposition 2.4. *Let* $\underline{x} = x_1, \ldots, x_r$ *be elements in* $\mathfrak{m}$. *For* $i = 1, \ldots, r$, *set* $u_i = (x_i, 0)$ *and* $\underline{u} = u_1, \ldots, u_r$. *The statements below are equivalent.*

(a) $\underline{u}$ *is a good system of parameters of* A.

(b) $\underline{x}$ *is a good system of parameters of* R *and* $x_1, \ldots, x_d$ *is a good system of parameters of* M. *Furthermore, if* $d < r$, *then* $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R M$.

Corollary 2.5. *There always exists a good system of parameters of* A *of the form* $(x_1, 0), \ldots, (x_r, 0)$*, where* $x_1, \ldots, x_r$ *is a good system of parameters of* R *and* $x_1, \ldots, x_d$ *is a good system of parameters of* M. *And if* $d < r$, *then* $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R M$.

3. Proof of main result

From now on, let $\mathfrak{F}_R : R_0 \subseteq R_1 \subseteq \ldots \subseteq R_r = R$ and $\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ be the dimension filtrations of R and M, respectively. Let $\underline{y} = x_1, \ldots, x_d$ be a good system of parameters of M. Then $x_1, \ldots, x_i$ is a multiplicity system of M_i, for all $i = 0, \ldots, d$. Therefore, the following function is well-defined

$$I_{\mathfrak{F}_M}(\underline{y}) = \ell(M / \underline{y}M) - \sum_{i=0}^{d} e(x_1, \ldots, x_i; M_i),$$

where $e(x_1, \ldots, x_i; M_i)$ is the multiplicity symbol of M_i with respect to $x_1, \ldots, x_i$, for $i = 0, 1, \ldots, d$. It is clear that $e(x_1, \ldots, x_i; M_i) = 0$ if and only if $\dim M_i < i$. Thus, the above concept of $I_{\mathfrak{F}_M}(\underline{y})$ is identical to the concept of $I_{\mathfrak{F}_M}(\underline{y})$ of N.T. Cuong et al. in [17]. However, for the convenience of calculations, we will use the above definition of $I_{\mathfrak{F}_M}(\underline{y})$. For any integers $\underline{m} = m_1, \ldots, m_d$, we denote

$$I_{\mathfrak{F}_M}(\underline{y}(\underline{m})) = \ell(M / (\underline{y}(\underline{m}))M) - \sum_{i=0}^{d} m_1 \ldots m_i e(x_1, \ldots, x_i; M_i)$$

as a function on $m_1, \ldots, m_d$ where $\underline{y}(\underline{m}) = x_1^{m_1}, \ldots, x_d^{m_d}$. By [17, Lemma 2.7, Proposition 2.9], we have the following lemma.

Lemma 3.1. Let $\underline{y} = x_1, \ldots, x_d$ be a good system of parameters of M. Then the function $I_{\mathfrak{F}_M}(\underline{y}(\underline{m}))$ is non-decreasing and non-negative.

We set $A_i = R_i \times M_i$ for $i = 0, \ldots, d$ and $A_j = R_j \times M$ for $j = d+1, \ldots, r$. Following Lemma 2.2, $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_r$ is the dimension filtration of A.

Lemma 3.2. *Let* $\underline{x} = x_1, \ldots, x_r$ *be a good system of parameters of* R. *Set* $\underline{u} = u_1, \ldots, u_r$, *where* $u_i = (x_i, 0)$ *for* $i = 1, \ldots, r$. *Then* $\underline{u}$ *is a system of parameters of* A. *Moreover, if* $\underline{y} = x_1, \ldots, x_d$ *is a good system of parameters of* M *and* $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R M$, *then for any positive integers* $n_1, \ldots, n_r$, *we have*

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{y}(\underline{m})).$$

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. *(a)* $\Rightarrow$ *(b).* By Corollary 2.5, A admits a good system of parameters of the form $u_1 = (x_1, 0), \ldots, u_r = (x_r, 0)$. Following [18, Remark 2.3], $(x_1, 0)^{n_1}, \ldots, (x_r, 0)^{n_r}$ is a good system of parameters of A for all $n_1, \ldots, n_r \geq 1$. Since A is sequentially Cohen-Macaulay, we get by [18, Theorem 4.2] that $I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = 0$ for all positive integers $n_1, \ldots, n_r$, which implies that

$$\ell\left(A/(u_1^{n_1}, \ldots, u_r^{n_r})\right) = \sum_{i=0}^{r} n_1 \ldots n_i e(u_1, \ldots, u_i; A_i)$$

for all positive integers $n_1, \ldots, n_r$, where $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r$ is the dimension filtration of A and $u_1 = (x_1, 0), \ldots, u_r = (x_r, 0)$.

$(b) \Rightarrow (c)$. We consider into two cases.

The case $d = r$. Since $\underline{u} = (x_1, 0), \ldots, (x_r, 0)$ is a good system of parameters. Following [18, Remark 2.3], $\underline{u}(n) = (x_1^{n_1}, 0), \ldots, (x_r^{n_r}, 0)$ is a good system of parameters of A for all $n_1, \ldots, n_r \geq 1$, we get by Proposition 2.4 that $\underline{x}(\underline{n}) = x_1^{n_1}, \ldots, x_r^{n_r}$ is a good system of parameters of both R and M for all positive integers $n_1, \ldots, n_r$. By the assumption (a) and Lemma 3.1, and Lemma 3.2, we have

$$0 = I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{x}(\underline{n})) \geq 0$$

for all $n_1, \ldots, n_r \geq 1$. Therefore, $I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) = 0$ and $I_{\mathfrak{F}_M}(\underline{x}(\underline{n})) = 0$ for all positive integers $n_1, \ldots, n_r$. The statement follows.

The case $d < r$. By Proposition 2.4, we have $x_1^{n_1}, \ldots, x_r^{n_r}$ is a good system of parameters of R, $x_1^{n_1}, \ldots, x_d^{n_d}$ is a good system of parameters of M and $x_{d+1}, \ldots, x_r \in \mathrm{Ann}_R M$ for all positive integers $n_1, \ldots, n_r$. Put $\underline{x} = x_1, \ldots, x_r$ and $\underline{y} = x_1, \ldots, x_d$. By the assumption (a) and Lemma 3.1, and Lemma 3.2, we have

$$0 = I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{y}(\underline{m})) \geq 0$$

for all positive integers $n_1, \ldots, n_r$. Therefore, $I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) = 0$ and $I_{\mathfrak{F}_M}(\underline{y}(\underline{m})) = 0$ for all positive integers $n_1, \ldots, n_r$. Thus, the statement follows.

$(c) \Rightarrow (b)$. Suppose that the condition (i) is satisfied. Since $x_1, \ldots, x_r$ is a good system of parameters of both R and M, we get by Proposition 2.4 that $\underline{u} = (x_1, 0), \ldots, (x_r, 0)$ is a good system of parameters of A. According to the assumption (i) and by Lemma 3.2, we have

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{x}(\underline{n})) = 0$$

implying that

$$\ell\left(A/(u_1^{n_1}, \ldots, u_r^{n_r})\right) = \sum_{i=0}^{r} n_1 \ldots n_i e(u_1, \ldots, u_i; A_i)$$

for all positive integers $n_1, \ldots, n_r$.

Suppose that the condition (ii) is satisfied. By Proposition 2.4 that $\underline{u}=(x_1,0),\ldots,(x_r,0)$ is a good system of parameters of A. Combining the assumption (ii) and by Lemma 3.2, we have

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n}))=I_{\mathfrak{F}_R}(\underline{x}(\underline{n}))+I_{\mathfrak{F}_M}(\underline{y}(\underline{m}))=0$$

implying that

$$\ell\left(A/(u_1^{n_1},\ldots,u_r^{n_r})\right)=\sum_{i=0}^{r}n_1\ldots n_i e(u_1,\ldots,u_i;A_i)$$

for all $n_1,\ldots,n_r\geq 1$.

$(b)\Rightarrow(a)$ is trivial by [18, Theorem 4.2].

From Theorem 1.1 and its proof, we immediately obtain the following corollary (also see [24, Corollary 2].

Corollary 3.3. The following assertions are equivalent:

(i) A is sequentially Cohen-Macaulay.

(ii) R and M are sequentially Cohen-Macaulay.

We end this paper with the following examples.

Example 3.4. Let $S=k[[X_1,X_2,X_3,X_4]]$ denote the formal power series ring in four variables over a field k. Set $R=S/\mathfrak{a}$ and $M=S/I$, where $I=(X_1X_2,X_1X_3,X_2X_3)$ and $\mathfrak{a}=(X_4)\cap I$. Let x, y, z, t denote the images of X_1,X_2,X_3,X_4 in R, respectively. Then $\dim R=3$, and the dimension filtration of R is given by

$$\mathfrak{F}_R:(0)=R_0=R_1\subsetneq R_2=(t)R\subsetneq R_3=R,$$

where R_2 is a Cohen-Macaulay module of dimension 2. We have $\dim M=2$, and the dimension filtration of M is

$$\mathfrak{F}_M:(0)=M_0=M_1\subsetneq M_2=M.$$

Define the elements $x_1=t+z, x_2=x+y+z,$ and $x_3=yz$.

Then $x_3\in \operatorname{Ann}_R(M)$ and

$$\ell(R/(x_1^{n_1},x_2^{n_2},x_3^{n_3}))=2n_1n_2n_3+3n_1n_2,$$

$$\ell(M/(x_1^{n_1},x_2^{n_2})M)=3n_1n_2,$$

for all $n_1,n_2,n_3\geq 1$. By [25, Definition 2.1] and [18, Corollary 3.7], $\underline{x}$ is a good system of parameters of R and $\underline{y}$ is a good system of parameters of M. Therefore, we have

$$I_{\mathfrak{F}_R}(\underline{x}(\underline{n}))=\ell(R/(x_1^{n_1},x_2^{n_2},x_3^{n_3})R)-n_1n_2n_3e(\underline{x},R)-n_1n_2e(\underline{y};R_1)=0,$$

$$I_{\mathfrak{F}_M}(\underline{y}(\underline{n}))=\ell(M/(x_1^{n_1},x_2^{n_2})M)-n_1n_2e(\underline{y};M)=0,$$

for all $n_1,n_2,n_3\geq 1$. By Theorem 1.1(c), $R\ltimes M$ is sequentially Cohen-Macaulay.

Example 3.5. Let $S = k[[X,Y,Z,T]]$ denote the formal power series ring in four variables over a field k. Put $R = S/\mathfrak{a}$ and $M = S/I$, where $I = (X,Y,Z)$ and $\mathfrak{a} = (X,Y,Z) \cap (T)$. Let x, y, z, t denote the images of X, Y, Z, T in R, respectively. Then the dimension filtration of R is

$$\mathfrak{F}_R : (0) = R_0 = R_1 \subsetneq R_2 = (t)R \subsetneq R_3 = R,$$

We have R is a sequentially Cohen-Macaulay ring with $\dim R = 3$ and M is a Cohen-Macaulay module. By Corollary 3.3, $R \ltimes M$ is sequentially Cohen-Macaulay.

4. Conclusions

In this paper, we have studied the sequential Cohen-Macaulayness of the idealization $R \ltimes M$ of a finitely generated module M over a Noetherian local ring R. By examining the length function of $R \ltimes M$ with respect to a good system of parameters of the form $(x_1,0),\ldots,(x_r,0)$, we obtained a necessary and sufficient condition for the idealization to be sequentially Cohen-Macaulay. These results offer insight into the structure of idealizations and contribute to the broader understanding of sequential Cohen-Macaulay modules and their invariants.

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