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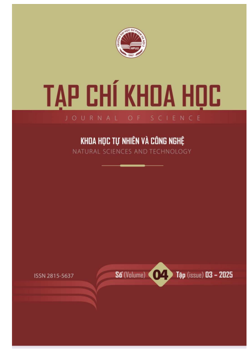

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Phase transitions and thermodynamic properties of the reissner-nordström black hole in (n + 1) dimensional spacetime

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Abstract

This study investigates the thermodynamic properties and phase transitions of charged Reissner-Nordström Anti-de Sitter black holes in $(n+1)$ -dimensional spacetime. The analysis is conducted within the framework of extended phase space thermodynamics, where the cosmological constant is treated as a dynamic pressure. We first derived analytical expressions for key thermodynamic quantities, then performed numerical calculations to explore the phase structure. Our analysis confirms a critical point and a first-order, liquid-gas-like phase transition analogous to the van der Waals system. A key finding is that the spatial dimension n significantly influences the critical parameters; black holes in higher dimensions exhibit higher temperatures and pressures in corresponding states. This work extends the well-known four-dimensional analogy to arbitrary dimensions, offering a more generalized perspective on black hole thermodynamics that is crucial for testing the universality of these phenomena and for higher-dimensional models in fundamental physics.

Keywords: Black hole thermodynamics, phase transitions, higher dimensions, critical point, van der Waals analogy

1. Introduction

The study of the thermodynamic properties of black holes, originating from the pioneering works of Bekenstein [1] and Hawking [2], has opened up one of the most profound avenues in theoretical physics, connecting gravity, quantum mechanics, and thermodynamics. These discoveries revealed that black holes are not static entities but possess thermodynamic attributes such as entropy and temperature. This analogy is not merely a formal coincidence but suggests a deeper microscopic structure of spacetime.

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The advent of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [3], [4] provided a powerful tool for probing the quantum aspects of gravity through a dual, non-gravitational field theory. In this context, black holes in AdS spacetime play a central role, and their thermodynamic properties, like the Hawking-Page phase transition [5], have direct interpretations in the dual field theory. Early work on charged AdS black holes also highlighted their rich phase structures and potential for catastrophic events [6].

Recently, this field has been reinvigorated by the development of "extended thermodynamics," where the cosmological constant (Λ) is treated as a dynamical pressure [7], [8]. This approach has significantly enriched the phase space of AdS black holes, leading to the subfield of "black hole chemistry" [9]. A key discovery is that charged Reissner-Nordström-AdS (RN-AdS) black holes exhibit a perfect analogy with the van der Waals liquid-gas system, complete with a critical point and phase transitions [10]. This analogy has been deepened by exploring concepts like Maxwell's equal area law [11], the existence of triple points [12], and extensions to rotating black holes and alternative theories like Born-Infeld [13].

Previous research has primarily analyzed these properties in four-dimensional spacetime [14]. However, a fundamental question remains: How do these thermodynamic structures depend on the spacetime dimension? Generalizing these studies to $(n+1)$ dimensions is crucial for testing the universality of these phenomena and for higher-dimensional models like string theory. Investigations into higher dimensions have already begun, exploring $P-V$ criticality in various contexts, including nonlinear sources [15] and Gauss-Bonnet gravity [16], [17]. Further thermodynamic behaviors, such as Joule-Thomson expansion [18] and the formulation of holographic heat engines [19], have also been explored, alongside analyses using thermodynamic geometry [20].

In this work, we perform a comprehensive analysis of the thermodynamics of Reissner-Nordström-AdS black holes in $(n+1)$-dimensional spacetime to systematically investigate their properties and explore how the dimension n affects the critical points and phase transition behaviors.

2. Methodology and Formalism

We begin with the RN charged black hole in $(n+1)$ dimensional Anti-de Sitter (AdS_{n+1}) spacetime whose metric is given by

$$ds_{n+1}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2. \tag{1}$$

in which

$$f(r) = k - \frac{8\Gamma\left(\frac{n}{2}\right)MG_{n+1}}{(n-1)\pi^{\frac{n}{2}-1}r^{n-2}} + \frac{Q^2}{r^{2n-4}} + \frac{r^2}{L^2}. \tag{2}$$

Here, M and Q are the mass and total charge of the black hole, respectively; L is the AdS radius (related to the cosmological constant $\Lambda : \Lambda = -\frac{n(n-1)}{2L^2}$) and $d\Omega_{n-1}^2$ is the metric of the $n-1$ dimensional base manifold; k is the spatial curvature of the black hole. Specifically, $k>0, k=0$ and $k<0$ correspond to spherical, flat, and hyperbolic symmetries, respectively.

By definition, the radius of the event horizon r_h is the largest positive root of $f(r_h)=0$. So

$$f(r_h)=k-\frac{8\Gamma\left(\frac{n}{2}\right)MG_{n+1}}{(n-1)\pi^{\frac{n}{2}-1}r_h^{n-2}}+\frac{Q^2}{r_h^{2n-4}}+\frac{r_h^2}{L^2}. \tag{3}$$

From this, it follows that the mass M can be expressed as:

$$M=\frac{(n-1)\pi^{\frac{n}{2}-1}}{8G_{(n+1)}\Gamma\left(\frac{n}{2}\right)}\left(kr_h^{n-2}+\frac{Q^2}{r_h^{n-2}}+\frac{r_h^n}{L^2}\right). \tag{4}$$

Inverting (4) into (2) we obtain

$$f(r)=k+\frac{r^2}{L^2}+Q^2r^{4-2n}-r^{2-n}\left(Q^2r_h^{2-n}+kr_h^{-2+n}+\frac{r_h^n}{L^2}\right). \tag{5}$$

The Hawking temperature T of the black hole is determined by the expression [4]

$$T=\frac{f'(r_h)}{4\pi}. \tag{6}$$

Combining the equations above, we obtain:

$$T=\frac{1}{4\pi}\left[(n-2)kr_h^{-1}+\frac{n}{L^2}r_h^{+1}-\frac{(n-2)Q^2}{r_h^{2n-3}}\right]. \tag{7}$$

For a charged RN black hole, the pressure is determined by [7]

$$P=-\frac{\Lambda}{8\pi}=\frac{(-1+n)n}{16L^2\pi}. \tag{8}$$

and the volume is determined by

$$V=\frac{\Omega_{n-1}}{n}r_h^n. \tag{9}$$

The entropy of the black hole is determined by

$$S=\frac{\Omega_{n-1}}{4}r_h^{n-1}. \tag{10}$$

3. Results and Discussion

In this section, we perform numerical calculations to clarify the thermodynamic properties of the black holes. For computational convenience, we will use dimensionless quantities below.

First, let's consider the equation of state $P(V,T)$. Here, T_c denotes the critical temperature, S_c the critical entropy, and V_c the critical volume.

$$T_c = \frac{\sqrt{\frac{k}{Q^2}}\left(k\left(-2+n\right)-6^{\frac{1}{2}(4-2n)}\left(-2+n\right)\left(-1+n\right)\left(\frac{k}{Q^2}\right)^{\frac{1}{2}(-4+2n)}Q^2\right)}{2\sqrt{6}\pi},$$

$$V_c = \frac{\left(2^{\frac{1}{4-2n}}\left(\frac{k}{\left(6-10n+4n^2\right)Q^2}\right)^{\frac{1}{4-2n}}\right)^n \Omega_{-1+n}}{n},$$

$$P_c = x_1 + x_2 + x_3. \tag{11}$$

in which

$$x_1 = -\frac{k^2\left(-2+n\right)\left(-1+n\right)}{96\pi Q^2};$$

$$x_2 = \frac{2^{-4+\frac{1}{2}(2-2n)}3^{\frac{1}{2}(2-2n)}\left(-2+n\right)\left(-1+n\right)\left(\frac{k}{Q^2}\right)^{\frac{1}{2}(-2+2n)}Q^2}{\pi};$$

$$x_3 = \frac{k\left(-1+n\right)\left(k\left(-2+n\right)-6^{\frac{1}{2}(4-2n)}\left(-2+n\right)\left(-1+n\right)\left(\frac{k}{Q^2}\right)^{\frac{1}{2}(-4+2n)}Q^2\right)}{48\pi Q^2}.$$

and combining equations (7), (8), and (9), we can write

$$\frac{P}{P_c} = \frac{2(y_1+y_2+y_3)}{-36k^{\frac{6+n^2}{-2+n}}\left(-3+2n\right)Q^{\frac{10n}{-2+n}} + 6^n k^{\frac{5n}{-2+n}} Q^{\frac{2\left(6+n^2\right)}{-2+n}}}. \tag{12}$$

in which

$$y_1 = -36\sqrt{6}k^{\frac{15-n+2n^2}{2(-2+n)}}\left(-1+n\right)\left(3-5n+2n^2\right)^{\frac{1}{4-2n}}Q^{\frac{-3+11n}{-2+n}}\frac{T}{T_c}\frac{V}{V_c}^{-1/n};$$

$$y_2 = 6^{\frac{1}{2}+n}k^{\frac{3(1+3n)}{2(-2+n)}}\left(3-5n+2n^2\right)^{\frac{1}{4-2n}}Q^{\frac{9+n+2n^2}{-2+n}}\frac{T}{T_c}\frac{V}{V_c}^{-1/n};$$

$$y_3 = \frac{2^n 3^{1+n}k^{\frac{3+4n}{-2+n}}\left(3-5n+2n^2\right)^{\frac{1-n}{-2+n}}Q^{\frac{2\left(3+n+n^2\right)}{-2+n}}\left(-3+5n-2n^2+\frac{V}{V_c}^{2/n}\right)}{\frac{V}{V_c}^{2}}.$$

To check the calculations, we have re-checked equations (11) for the case $n = 3$ and the results obtained are completely consistent with the results found in the document [14], which proves that our calculations are completely correct.

Based on the equation of state, we can draw the dependence of the pressure P on the volume V at several values of temperature T. To clearly illustrate the role of temperature, we select values of T/T_c that are greater than 1, equal to 1, and less than 1, corresponding to temperatures greater than, equal to, and less than the critical temperature T_c. Figure 1 shows the form of several isotherms in the $P-V$ plane. They exhibit a behavior similar to the isotherms of a real van der Waals gas. Furthermore, for temperature $T<T_c$ the isotherm exhibits non-monotonic behavior. This indicates that when $T<T_c$ the matter can exist in either a “liquid” or “gas” phase, and a “liquid-gas” type phase transition, analogous to a real van der Waals gas, occurs as the volume changes. Conversely, for $T>T_c$ there is no phase transition, and the matter exists only in a single phase. At $T=T_c$,the isotherm has an inflection point, which confirms that $T=T_c$ is indeed the critical temperature. It's important to emphasize that the terms “gas” and “liquid” are used here solely due to the analogy with the van der Waals gas.

The solid lines ($n=3$) and dashed lines ($n=5$) illustrate a clear difference. Specifically, in higher-dimensional spacetime ($n=5$), the pressure tends to be slightly higher at the same relative volume and temperature. This implies that the “interaction forces” between the black hole's microscopic constituents might be stronger in higher dimensions, or that the structure of the quantum vacuum changes with dimensionality.

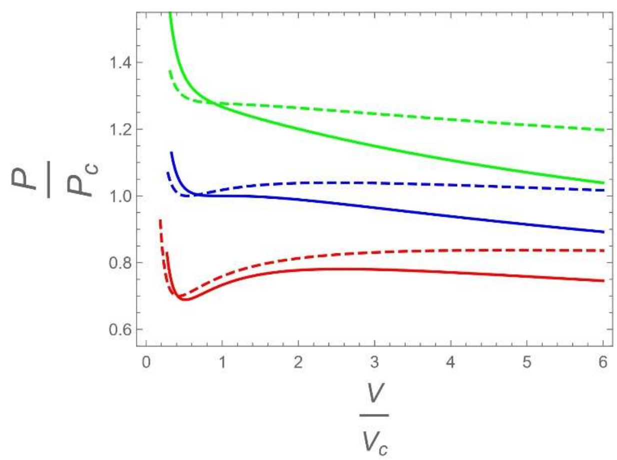

Figure 1. Pressure P as a function of volume V.

Next, we investigate the dependence of the Hawking temperature on the horizon radius (which is also equivalent to the black hole's volume). Based on equations (7) and (8), we can write

$$\frac{T}{T_c}=\frac{2^{-\frac{3}{2}+n}3^{-\frac{1}{2}+n}\left(3-5n+2n^2\right)^{\frac{3-2n}{2(-2+n)}}\frac{r_h}{r_c}^{3-2n}(z_1+z_2)}{6^n k^{\frac{-3+n}{2(-2+n)}}-36k^{\frac{9-9n+2n^2}{2(-2+n)}}\left(-1+n\right)Q^{6-2n}}. \tag{13}$$

in which

$$z_1=6^{-n}\left(3-5n+2n^2\right)^{\frac{-1+n}{-2+n}}\frac{P}{P_c}\left(-36k^{n-3}\left(-3+2n\right)Q^{\frac{-9+9n-2n^2}{-2+n}}+6^n Q^{\frac{-n+3}{-2+n}}\right)\frac{r_h}{r_c}^{-2+2n};$$

$$z_2=6Q^{\frac{-3+n}{-2+n}}\left(-1+\left(3-5n+2n^2\right)\frac{r_h}{r_c}^{-4+2n}\right).$$

where

$$r_c = \sqrt[4-2n]{\frac{2k}{Q^2\left(4n^2-10n+6\right)}}. \tag{14}$$

is the critical horizon radius. The numerical result in Figure 2. Here, the selection of the ratio values P/P_c are also intended to clarify the role of pressure around the critical value P_c similar to the T/T_c selection mentioned earlier.

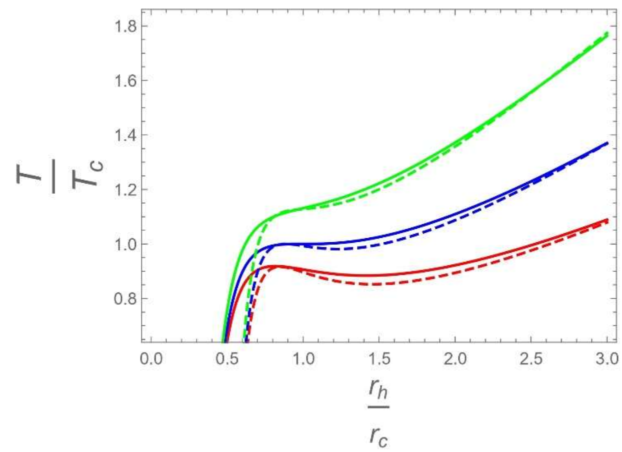

Figure 2. Hawking temperature T as a function of horizon radius r_h at several pressure values.

Using (7), (8) and (10), we can write

$$\frac{T}{T_c} = \frac{h_1+h_2+h_3}{2\sqrt{6}\left(6^n k^{\frac{11-3n}{4-2n}} - 36k^{\frac{1-7n+2n^2}{2(-2+n)}}\left(-1+n\right)Q^{6-2n}\right)}. \tag{14}$$

in which

$$h_1 = 6^{1+n} k^{\frac{-4+n}{-2+n}} \left(3-5n+2n^2\right)^{\frac{1}{4-2n}} Q^{\frac{-3+n}{-2+n}} \frac{S}{S_c}^{\frac{1}{1-n}};$$

$$h_2 = k^{\frac{1+n}{2-n}} \left(3-5n+2n^2\right)^{\frac{1}{2(-2+n)}} \frac{P}{P_c} Q^{\frac{3+3n-2n^2}{-2+n}} \left(-36k^n\left(-3+2n\right)Q^6 + 6^n k^3 Q^{2n}\right) \frac{S}{S_c}^{\frac{1}{-1+n}};$$

$$h_3 = -6^{1+n} k^{\frac{-4+n}{-2+n}} \left(3-5n+2n^2\right)^{\frac{3-2n}{2(-2+n)}} Q^{\frac{-3+n}{-2+n}} \frac{S}{S_c}^{\frac{3-2n}{-1+n}}.$$

where the critical entropy S_c is determined by

$$S_c = 2^{\frac{2n-4}{2-n}} \left(\frac{k}{\left(3-5n+2n^2\right)Q^2}\right)^{\frac{-1+n}{4-2n}} \Omega_{-1+n}. \tag{15}$$

Based on (15) we can draw the entropy S dependence of the temperature T. Figure 3 illustrates the form of the $T(S)$ curve at several specific pressure values.

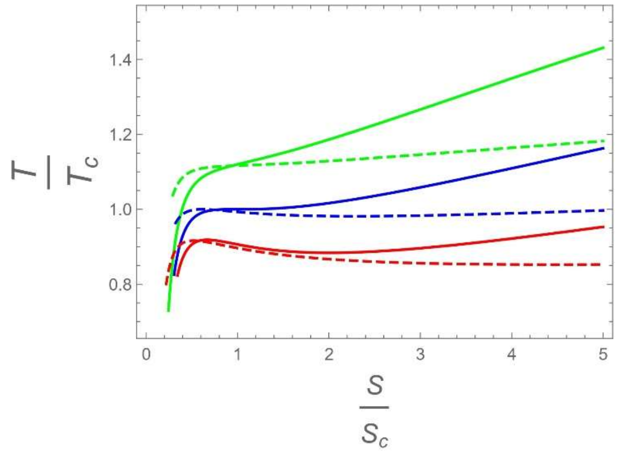

Figure 3. Temperature T as a function of entropy S at several pressure values.

Figures 2 and 3 offer a different perspective on the same phase transition phenomenon. The isobars $(p = \text{const})$ also exhibit non-monotonic behavior when $P < P_c$.

The region with a negative slope ($dr_h / dT < 0$ or $dS / dT < 0$) corresponds to a thermodynamically unstable phase (negative specific heat). A black hole in this state will either shrink to a smaller black hole (liquid phase) or expand into a larger black hole (gas phase) to reach a more stable state.

Similar to Figure 1, we observe that for the same relative pressure, the black hole's temperature in 5-dimensional spacetime tends to be higher. This indicates that black holes are "hotter" in higher dimensions. The positions of the extremal points (corresponding to the boundaries of the unstable region) also shift, demonstrating a strong dependence of the critical parameters (P_c, V_c, T_c) on the n -dimensional.

4. Conclusion

In this paper, we extended the study of thermodynamics and phase transitions of charged Reissner-Nordström-AdS black holes to ($n+1$)-dimensional spacetime. Our comprehensive analysis within the framework of extended phase space thermodynamics has yielded several key findings:

- We derived analytical expressions for the fundamental thermodynamic quantities, including temperature, entropy, and the equation of state, applicable to arbitrary dimensions.

- The numerical analysis unequivocally confirmed the existence of a liquid-gas-like phase transition, analogous to the van der Waals system, which is a crucial characteristic preserved across different dimensions. This result further reinforces the concept of black holes as complex thermodynamic systems.

- The most significant new contribution of this study is the detailed clarification of how the spatial dimension n profoundly influences the critical parameters and the overall phase transition behavior. Specifically, by increasing the dimension from 4 to 5, we observed that black holes tend to be "hotter" and exhibit higher pressures in corresponding states. This observation suggests that the spatial dimension is an important physical parameter, potentially adjusting the "interaction constants" within an unknown microscopic theory of quantum gravity. These findings collectively offer a more generalized and comprehensive perspective on black hole thermodynamics within higher-dimensional spacetimes.

For a more comprehensive understanding, future research will explore cases with different spatial curvatures ($k = -1$ and $k = 0$), as well as investigate more complex types of black holes. This will be the focus of our future research endeavors.

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