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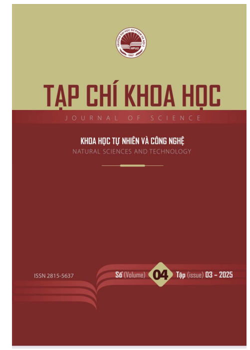

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On the spectral radius of bipartite graphs with bounded partite sets

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Abstract

This paper investigates the relationship between spectral graph theory and graph properties, specifically focusing on the spectral radius, which is the largest eigenvalue of the adjacency matrix of a graph. Our problem is finding sharp upper bounds for the spectral radius of bipartite graphs with given bounde vertex sets. We first review existing inequalities, noting and discuss their limitations, noting particularly that some, like Hong's inequality, are not always sharp for all graph types. Our primary contribution is an elementary and direct approach to solving an optimization problem: finding a bipartite graph with a bounded number of vertices that maximizes its spectral radius. We prove that for any bipartite graph with bounded vertex sets of n_1 and n_2, the spectral radius $\rho(G)$ is bounded by $\sqrt{n_1 n_2}$. We demonstrate that this inequality is sharp, with equality holding exclusively for the complete bipartite graph K_{n_1,n_2}.

Keywords: Spectral graph theory, spectral radius, upper bounds, bipartite graphs, eigenvalues, adjacency matrix, graph optimization

1. Introduction

Spectral graph theory studies the relation between graph properties and the spectrum of the adjacency matrix or Laplacian matrix. Spectral graph theory appeared in the 1950s and they rapidly found some applications in quantum chemistry [1] and complex networks [2]–[6]. Spectral radius, or the largest eigenvalue of the adjacency matrix of graph G appears in many applications [3]–[6], such as spreading viruses on complex networks [3]–[5]. It is also powerful tool to characterize dynamic processes on networks [3]–[5] and investigate the Cheeger inequality in Riemannian geometry [2].

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The problem on estimating or finding good upper bounds for the spectral radius of a graph is an important topic. It is related to some models of virus spreading in networks [4], biogogical networks and random graphs [2], [5]. One of the first estimate is of Wilf [7] and Brualdi-Hoffman [8] for a simple connected graph with n vertices and m edges, then the spectral radius $\rho(G)$ of the graph G satisfies $\chi(G)-1 \le \rho(G) \le k-1$, where $m=\frac{k(k-1)}{2}$ and $\chi(G)$ is the chromatic number of graph G. Another upper bound is of Stanley [9], that is $\rho(G) \le \frac{\sqrt{8m+1}-1}{2}$. In [10], Hong gave an unfamous upper bound on the spectral radius of a graph. More explicitly, the author proved that if G is a simple connected graph then

$$\rho(G) \le \sqrt{2m-n+1}, \tag{1}$$

We call (1) Hong's inequality. After that, many upper bounds are obtained in many cases of G, for instance, the case of K_{p+1}-free [11], or case of removing some vertices [12]–[14]. Due to the inequality (1), we have $\rho(G) \le \sqrt{2m}$.

In [11], Nikiforov extended the result of Nosal [15]. More explicitly, the following inequality

$$\rho(G) \le \sqrt{m}. \tag{2}$$

satisfies if G is K_{p+1}-free. It is easy to see that inequality (2) is sharper than both the inequality (1) and Stanley's one.

In this paper, we formulate and solve the optimization problem for the spectral radius of bipartite graphs with bounded numbers of vertices. If we apply the inequality (2), we can see a simpler solution to our problem. Here, however, we present another approach. Our approach is more elementary than Nosal-Nikoforov theorem, as it is based on some very simple observations. In particular, we use inequalities on a graph after deleting a vertex [12]–[13]. Besides, we remark the sharpness on the bounds of Hong and Nosal for the complete bipartite and complete graphs respectively.

The rest of paper is organized as follows. In Section 2, we recall some notions in graph theory, some properties of spectral radius and some known properties of bipartite graphs. The main results are in Section 3, where we give the answer for Problem 3.2 in Theorem 3.3.

2. Preliminaries

We present notions and preliminary results from graph theory and spectral graph theory, following references [12]–[14].

2.1. Some notions in graph theory

In this paper, we consider finite undirected, and simple graphs, i.e. $G=(V,E)$ is an undirected graph with V is the finite vertex set, E is the finite edge set and G has not loops or multiple edges.

Definition 2.1. Let $G=(V,E)$ be a finite undirected simple graph. Suppose that members of V are labelled $1,2,\dots,n$. If vertices i and j are joined by an edge, then we say that i and j are *adjacent* and write $i \sim j$. We define the *adjacency matrix* A of G as follows: $A = A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 \; i \sim j, \\ 0 \text{ otherwise.} \end{cases}$$

Definition 2.2. Let $G=(V,E)$ be a finite undirected simple graph. A graph $H=(U,D)$ is called a *subgraph* of graph G, written $H \subseteq G$ if $U \subseteq V$ and $D \subseteq E$.

Definition 2.3. Let $G=(V,E)$ be a finite undirected simple graph. The *characteristic polynomial of graph* G is $\det(xI-A)$. The *eigenvalues of graph* G are the eigenvalues of the adjacency matrix $A(G)$, i.e. roots of the characteristic polynomial of G and the set of the eigenvalues of G is said to be the spectrum of G. We denote spectrum of G by $Sp(G)$.

Remark 2.4. Since the above definition of adjacency matrix, $A(G)$ is a symmetric matrix. Hence, all of eigenvalues of G are real numbers.

Definition 2.5. Deletion of an edge e from a graph $G=(V,E)$ is the operation that removes e from E and results in the subgraph $G-\{e\}=(V,E\setminus\{e\})$ of G.

Remark 2.6. Note that after deleting an edge from a graph, $|E'|=|E|-1$ but $|V'|=|V|$, i.e., the subgraph $G-\{e\}$ keeps all of vertices of G.

Definition 2.7. Deletion of a vertex j from a graph $G=(V,E)$ is the operation that excludes j from V and all edges with endpoint j from E. The resulting subgraph of G is denoted by $G-\{j\}$.

Definition 2.8. Let $G=(V,E)$ be simple undirected graph. G is called *bipartite graph* if the set V can be partitioned into two disjoint subsets V_1 and V_2, called *partite sets* such that every edge of G joins a vertex of V_1 and a vertex of V_2.

From the definition of bipartite graph, we have: $V=V_1\cup V_2, V_1\cap V_2=\varnothing$ and there is no edge joins the vertex of $V_i (i=1,2)$ and a vertex of itself.

Definition 2.9. A simple undirected graph $G=(V,E)$ is *complete* if every pair of distinct vertices in V is connected by a unique edge in E.

Definition 2.10. A *complete bipartite graph* is a bipartite graph with partite sets V_1, V_2 such that for any $i\in V_1$ and $j \in V_2$, we have $(i,j)\in E$. We denote the complete bipartite graph by $K_{m,n}$ when $|V_1|=m, |V_2|=n$.

2.2. Some properties of spectral radius

We recall that the largest eigenvalues of the adjacency of graph G is called the spectral radius of G

Lemma 2.12 ([12], p.17). If $G-\{ij\}$ is the graph obtained from a connected graph G by deleting the edge ij, then $\rho(G-\{ij\}) \le \rho(G)$.

Proof. (cf. [12]) Let $v=(x_1,\ldots,x_n)^T$ be a nonnegative eigenvetor of $G-\{ij\}$ corresponding to $\rho(G-\{ij\})$ which is a unit vector. Then we have

$$\rho(G-\{ij\}) = v^T A(G-\{ij\})v \le v^T A(G)v \le \rho(G).$$

The lemma is proved.

Lemma 2.13 ([12], p.17). If $G-\{j\}$ is the graph obtained from a connected graph G by deleting the vertex j, then $\rho(G-\{j\}) \le \rho(G)$.

Proof. (cf. [12]) Let A, A' be adjacency matrices of G, $G-j$, respectively, then $A = \begin{bmatrix} A' & r \\ r & 0 \end{bmatrix}$. Let v be a unit eigenvector of A' corresponding to $\rho(G-j)$. Take $u = \begin{bmatrix} v \\ 0 \end{bmatrix}$, then $u^T u = 1$ and therefore $\rho(G-\{j\}) = u^T Au \le \rho(G)$. The lemma is proved.

Proposition 2.14 ([12]-[14]). *The number of closed walks of length k in a graph G is equal to s_k, where*

$$s_k = \sum_{i=1}^{n} \lambda_i^k,$$

and $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ are the eigenvalues of G. Hence, the number of edges of G is $\frac{s_2}{2}$ and the number of triangles in G is $\frac{s_3}{6}$.

Theorem 2.15. (Hong, [10]) *Let G be a connected simple graph with m edges and n vertices. Then the spectral radius of the adjacency matrix A of graph G satisfies*

$$\rho(G) \le \sqrt{2m-n+1}$$

with equality if and only if G is isomorphic to one of the following two graphs:

a. the star $K_{1,n-1}$.

b. *the complete graph K_n.*

In the master thesis [15], E. Nosal gave an upper bound for the spectral radius in the case of triangle-free. After that, Nikiforov [11] extended for the graph which is K_{p+1}-free.

Theorem 2.16 (Nosal-Nikiforov, [11], [15]) *Let G be a graph with $\omega(G) \le p$ (then G is K_{p+1}-free). Then*

$$\rho(G) \le \sqrt{\frac{p-1}{p} 2m},$$

where $\omega(G)$ is the clique number of G.

2.3. Some properties of bipartite graphs

Here, we recall the following characterizations of bipartite graphs ([14], [16]).

Theorem 2.17. *The following statements are equivalent for a graph G:*

i. G is bipartite;

ii. G *has no cycle of odd length;*

iii. The characteristic polynomial $p(\lambda)=\sum_{i=0}^{n}c_i\lambda^{n-i}$ *of* $A(G)$ *satisfies* $c_k=0$ *for each odd integer* k*;*

iiii. $Sp(G)=-Sp(G)$.

Lemma 2.18. ([14], 1.4.2) *The eigenvalues of complete bipartite graph* K_{n_1,n_2} *are* 0 *(with multiplicity* n_1+n_2-2*),* $\sqrt{n_1n_2}$ *(with multiplicity* 1*),* $-\sqrt{n_1n_2}$ *(with multiplicity* 1*). Hence, the spectral radius of* K_{n_1,n_2} *is* $\sqrt{n_1n_2}$.

Proposition 2.19. ([14], 1.4.1) *The eigenvalues of complete graph* K_n *are* $n-1$ *(with multiplicity* 1*) and* -1 *(with multiplicity* $n-1$*).*

3. Problem and main results

3.1. On the upper bounds of the spectral radius of Hong

We give the following remark to show that Hong's inequality (1) is not sharp in the case of graph G , which is a complete bipartite graph. Howerver, it is sharp for complete graph.

Proposition 3.1. *The above upper bound of* $\rho(G)$ *in Hong's inequality is not sharp for complete bipartite graph* $K_{n,n}$*, where* $n>1$*. The upper bound of Hong is sharp for complete graph* K_n .

Proof. Consider complete bipartite graph $K_{n,n}$, by Lemma 2.18, the spectrum of $K_{n,n}$ is $Spec(K_{n,n})=\{0,n,-n\}$. Then the spectral radius of $K_{n,n}$ is $\rho(K_{n,n})=n$. By substituting this into (1), we have $n\le\sqrt{2n^2-2n+1}$. This is equivalent to $0\le n^2-2n+1$. It is easy to see that this inequality is not sharp with n is an integer and $n\ge 4$ (because $n=1,2,3$ are the cases of trivial complete bipartite graphs). Therefore, in the inequality (1), we can repair the right-hand side to get the sharper inequality.

Let us consider complete graph K_n , by Proposition 2.19, $\rho(K_n)=n-1$. From Hong's inequality, we have $n-1\le\sqrt{2\frac{n(n-1)}{2}-n+1}$. This is an equality, hence the upper bound of (1) is sharp for K_n .

3.2. An upper bound for spectral radius of bipartite graphs

By Proposition 3.1, we have proven that the upper bound of Hong [10] is not sharp and it is weaker than the bound of Nosal-Nikiforov [11], [15]. Consequently, this is our motivation to propose the following problem.

Problem 3.2. Find a bipartite graph $G=(V_1\cup V_2,E)$ with $|V_1|\le n_1\,|V_2|\le n_2$ such that of the spectral radius of G gets the maximum values.

Let $Bip(n_1,n_2)$ be the set of all bipartite graphs which have two sets of vertices V_1,V_2 such that

$|V_1| \le n_1 \ |V_2| \le n_2$. Then the problem becomes

$$\text{Find } \max_{G \in Bip(n_1,n_2)} \rho(G).$$

The following result provides the answer for the problem. Here, we present an upper bound for a connected bipartite graph. Our proof is very brief and are based on two simple observations.

Theorem 3.3. *Let $G=(V,E)$ be a bipartite graph with $V=V_1 \cup V_2$ (disjoint), where $|V_1| \le n_1, |V_2| \le n_2$. Then the spectral radius of G satisfies the following inequality.*

$$\max_{G \in Bip(n_1,n_2)} \rho(G) = \sqrt{n_1 n_2},$$

where n_1, n_2 are the numbers of the vertices of two parts of K_{n_1,n_2}. The inequality holds if and only if G is a complete bipartite graph K_{n_1,n_2}.

Proof. Let us consider a bipartite graph G, where $|V_1| \le n_1, |V_2| \le n_2$. Recall that $Bip(n_1,n_2)$ be the set of all bipartite graphs which have two sets of vertices V_1, V_2 such that $|V_1| \le n_1 \ |V_2| \le n_2$. The following claim is similar to Proposition 2.2.7 about k-regular graph in [16].

Claim 1. *Any graph $G \in Bip(n_1,n_2)$ can be obtained from K_{n_1,n_2} after deleting a finite number of edges and vertices. Moreover, we can see G can be viewed as subgraph of K_{n_1,n_2}.*

Proof of Claim 1.

Suppose that $G \in Bip(n_1,n_2)$, then G is a bipartite graph with $|V_1| \le n_1, |V_2| \le n_2$. Assume that K_{n_1,n_2} has the sets of vertices that are U_1, U_2 such that $|U_1| = n_1, |U_2| = n_2$ and $E(K_{n_1,n_2})$ is the edges set of K_{n_1,n_2}. We prove that G becomes K_{n_1,n_2} after adding some vertices and edges structurally. G can be obtained to K_{n_1,n_2} by the following procedure. It has two steps.

- Step 1: From G, we can consider V_i is a subset of U_i ($i=1,2$). Add $U_i \setminus V_i$ into the set V_i. Hence, we have the new graph $G' = (V_1' \cup V_2', E)$ with $|V_1'| = n_1, |V_2'| = n_2$. Graph G is a subgraph of graph G' and graph G' is a subgraph of K_{n_1,n_2}.
- Step 2: From G', we add $|E(K_{n_1,n_2})| - |E|$ edges into the set E from new edges in G'.

After doing the aforementioned two steps, K_{n_1,n_2} is obtained from G, hence proving the claim.

Now, return to the theorem. Using Claim 1, we can assume that after deleting k edges and l vertices of K_{n_1,n_2}, we obtain the graph G is obtained. By Lemma 2.13,

$$\rho(G) = \rho(G' - \{v_1, v_2, \ldots, v_{l-1}, v_l\}) \le \rho(G' - \{v_1, v_2, \ldots, v_{l-1}\}) \le \ldots \le \rho(G' - \{v_1\}) \le \rho(G'),$$

where G' is the bipartite graph which obtained after making deletion of l vertices $\{v_1, \ldots, v_l\}$. Now, using Lemma 2.12,

$$\rho(G') = \rho(K_{n_1,n_2} - \{e_1, e_2, \ldots, e_{k-1}, e_k\}) \le \rho(K_{n_1,n_2} - \{e_1, e_2, \ldots, e_{k-1}\}) \le \ldots \le \rho(K_{n_1,n_2} - \{e_1\}) \le \rho(K_{n_1,n_2}).$$

Since Lemma 2.18, the inequality $\rho(G) \le \sqrt{n_1 n_2}$ is obtained.

It is easy to see that the equality holds if and only if G is a complete bipartite graph K_{n_1,n_2}. The theorem is then completely proved.

Remark 3.4. Above problem can approach by Theorem 2.16 because the bipartite G has no cycle of odd length (Theorem 2.17). In Problem 3.2, G is a bipartite graph, then G is triangle-free, therefore we can apply Theorem 2.17. Our approach is more elemetary, as there is no need to apply the Nosal-Nikoforov theorem. By Remark 3.1, we can see that $\sqrt{n_1 n_2}$ is the sharpest bound for $\rho(G)$ on $G \in Bip(n_1, n_2)$. If G is the complete graph K_n, then the number of edges of G is $m = \frac{n(n-1)}{2}$. Hence

$$\rho(G) = n-1 = \sqrt{(n-1)^2} > \sqrt{n(n-1)} = \sqrt{2m}.$$

Consequently, the Nosal-Nikiforov's inequaltiy does not satisfy. The reason is that K_n contains some triangles when $n \ge 3$.

4. Conclusion

i. Here, we solved the combinatorial optimization problem which is finding the maximum of the spectral radius of a bibartite graph with bounded vertex sets.

ii. Recently, there are several works about the bounds of spectral radius in many cases [17]–[19]. More explicitly, the authors in [19] proved the conjecture in [18], that is

$$\rho(G) \le \sqrt{\rho^2(G - v_k) + 2d_k - 1},$$

where v_k is a vertex of graph G with the degree $d_k \ge 1$.

iii. The theory of graph spectra continues to find many powerful applications in complex networks. Recently, there have been applications in both physics and machine learning [20]–[21].

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