HPU2 Journal of Sciences:
Natural Sciences and Technology

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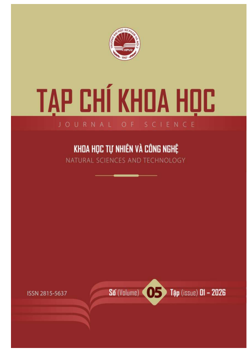

Article type: Research article

One-loop gauge boson contributions to decays $h \to e_a e_b$ in the general gauge R_ξ

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Abstract

Analytic formulas to express one-loop contributions to lepton flavor violating decays of the Standard Model-like Higgs boson $h \to e_a^{\pm} e_b^{\mp}$ are calculated in the general gauge R_ξ for the general seesaw version of the Standard Model. The calculation is performed using the Passarino-Veltman functions, which include various well-known relations useful for analytic transformations. This calculation also requires precise forms of the couplings of the Goldstone boson of $W^{\pm}$, which are model-dependent. We demonstrate analytically that the final result is gauge-independent, namely, the free unphysical parameter ξ of the R_ξ gauge vanishes completely. Moreover, the resulting decay amplitude is consistent with those previously calculated in different gauges, including the unitary gauge.

Keywords: Higgs boson, Beyond Standard Model, lepton flavor violating decays, one-loop contributions, neutrinos, etc

1. Introduction

The lepton-flavor-violating decays of the Standard Model-like Higgs boson (LFVh) are currently being searched by experiments at Large Hadron Collider (LHC) [1]–[3]. They are hoped to be clear signals of new physics beyond the prediction of the Standard Model (SM). Theoretically, LFVh decays were studied in many Beyond the SM (BSM) including the simple extensions of the SM with heavy neutral

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https://doi.org/10.56764/hpu2.jos.2025.5.1.3-13

Received date: 30-9-2025 ; Revised date: 03-11-2025 ; Accepted date: 30-3-2026

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leptons introduced to accommodate neutrino oscillation data. For models predicting that LFVh decays receive contributions from loop levels, the analytic formulas for the one-loop gauge-boson contributions were all calculated in particular gauges such as 't Hooft-Feynman and unitary. The two respective results were shown to be consistent, implying that the final amplitudes do not depend on the unphysical parameter that defines the gauge R_ξ appearing in the propagators of massive gauge bosons. In this work, the calculation will be performed in the general gauge R_ξ, focusing on a simple BSM with new neutral leptons that explain the neutrino oscillation data through the general seesaw (GSS) mechanism. To the best of our knowledge, this proof using analytical techniques has not been reported previously; therefore, our work is original and provides new specific relations between gauge bosons $W^\pm$ and their Goldstone bosons, which will be useful for further studies of BSM with complicated gauge boson spectra.

2. Detailed calculations

Detailed discussions on LFVh phenomenology related to the seesaw (SS) neutrino were given in many works [4]–[9]. The model we study in this work is a class of the SM extensions relating to the GSS mechanism [9], [10]. The Feynman rules of the couplings relating to LFVh are collected in Table 1

Vertex	Coupling	Vertex	Coupling
$hW^{+\mu}W^{-\nu}$	$igm_W g_{\mu\nu}$	$hG_W^+G_W^-$	$-\frac{igm_h^2}{2m_W}$
$hG_W^+W^{-\mu}$	$\frac{ig}{2}\left(p_+ - p_0\right)_\mu$	$hG_W^-W^{+\mu}$	$\frac{ig}{2}\left(p_0 - p_-\right)_\mu$
$\overline{n_i}e_aW_\mu^+$	$\frac{ig}{\sqrt{2}}U_{ai}^\nu\gamma^\mu P_L$	$\overline{e_a}n_iW_\mu^-$	$\frac{ig}{\sqrt{2}}U_{ai}^{\nu*}\gamma^\mu P_L$
$\overline{n_i}e_aG_W^+$	$-\frac{ig}{\sqrt{2}m_W}U_{ai}^\nu\left(m_aP_R - m_{n_i}P_L\right)$	$\overline{e_a}n_iG_W^-$	$-\frac{ig}{\sqrt{2}m_W}U_{ai}^{\nu*}\left(m_aP_L - m_{n_i}P_R\right)$
$h\overline{n_i}n_j$	$\frac{-ig}{2m_W}\left[\lambda_{ij}P_L + \lambda_{ij}^*P_R\right]$	$h\overline{e_a}e_a$	$-\frac{igm_a}{2m_W}$

Table 1. Couplings relating to LFVHD in seesaw models with λ_{ij} given in Eq. (1) and $p_{0,\pm}$ being incoming momenta of h, G_W^+ and G_W^-, respectively.

where the coupling $hG_W^+G_W^-$ is consistent with one given in Refs. [4], [5], [11]. Here we use the notation consistent with the standard Feynman rules introduced in Ref. [12] that

$$\lambda_{ij} = \lambda_{ji} = \sum_{d=1}^{3}\left(m_{n_i}U_{di}^\nu U_{dj}^{\nu*} + m_{n_j}U_{dj}^\nu U_{di}^{\nu*}\right), \tag{1}$$

and U^ν is the total mixing matrix that is defined precisely in Ref. [9] along with the analytic formula of the total mass matrix M^ν

The propagator of the gauge boson $W^\pm$ with mass m_W in the general gauge R_ξ is [13]

$$P_W^{(\xi)\mu\nu} = \frac{-i}{k^2 - m_W^2}\left(g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 - \xi m_W^2}\right), \tag{2}$$

which can be considered in the unitary gauge (u) as follows

$$\lim_{\xi\to\infty} P_W^{(\xi)\mu\nu} = P_W^{(u)\mu\nu}(k^2) = \frac{-i}{k^2-m_W^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}\right) = (-i)P_W'^{(u)\mu\nu}(k^2). \tag{3}$$

The formula of the propagator of the respective Goldstone boson $G_W^{\pm}$ absorbed by $W^{\pm}$ is

$$iP^{(\xi)}(k^2) \equiv P_{G_W}^{(\xi)}(k^2) = \frac{i}{k^2-\xi m_W^2} \xrightarrow{\xi\to\infty} P_{G_W}^{(u)}(k^2) = 0: \text{Unitary (u)}. \tag{4}$$

Therefore, contributions from diagrams including the Goldstone boson in the unitary gauge always vanish. The results of one-loop contributions to LFVh decay amplitudes in the two gauges, 't Hooft-Feynman and unitary, were given in Refs. [5], [6] and [9], respectively. The two total results, denoted as $\Delta_{L,R}$ in these gauges, are consistent with each other, as expected. Therefore, we do not repeat them in this work. On the other hand, in the unitary gauge, the gauge boson propagator has a special property that is useful for our calculation in the gauge R_ξ. Namely, the formula of the gauge boson propagator $P_W^{(\xi)\mu\nu}$ can be written in terms of two separated parts as follows [13]

$$P_W^{(\xi)\mu\nu}(k^2) \equiv P_W^{(u)\mu\nu}(k^2) + P_W^{(T)\mu\nu}(k^2),$$
$$P_W^{(T)\mu\nu}(k^2) = \frac{-i}{m_W^2}\times\frac{k^\mu k^\nu}{k^2-\xi m_W^2} = \frac{-k^\mu k^\nu}{m_W^2}P_{G_W}^{(\xi)}(k^2). \tag{5}$$

It is emphasized that the ξ dependent parts in the denominators of the two propagators of the $W^{\pm}$ and its Goldstone boson $G_W^{\pm}$ have the same form. As a result, this part is considered as the propagator of a particle with mass $m_{G_W}^2 = \xi m_W^2$ in our calculation. When summing contributions from all one-loop diagrams, the result will divide into two parts, in which the first is exactly the one derived from unitary gauge, which is completely independent of ξ. In contrast, the second consists of all terms with the same form of the ξ-dependency, suggesting a reasonable clue to ensure the zero value of this part with arbitrary ξ.

To express the effective coupling of the LFVh amplitude, the respective. The Lagrangian is written normally as $\mathcal{L}^{\mathrm{LFVh}} = h\overline{e_a}\left(\Delta_L^{(ab)}P_L + \Delta_R^{(ab)}P_R\right)e_b + \mathrm{h.c.}$, where $\Delta_{L,R}^{(ab)}$ are scalar factors derived from one-loop Feynman diagrams given in Figure 1.

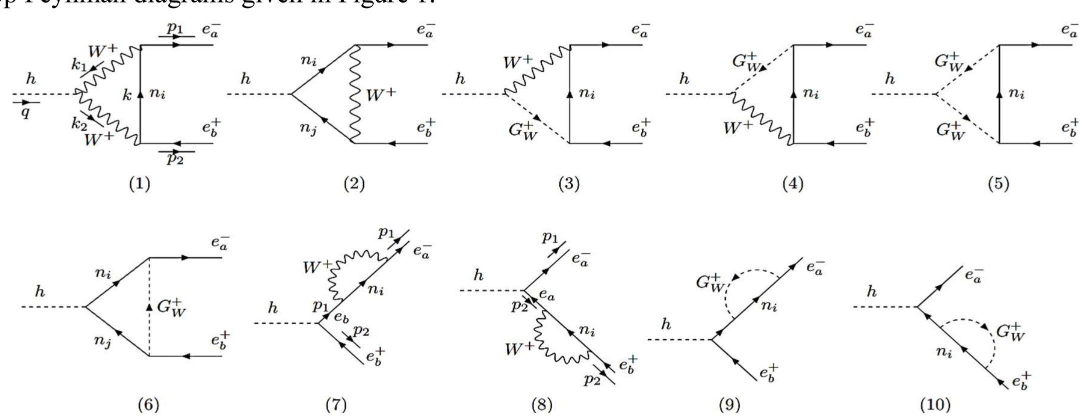

Figure 1. Feynman diagrams contributing to the LFVh decay amplitude in the general gauge R_ξ.

Limited in the SM extension adding GSS heavy neutrinos. Namely, using the Feynman rules, we will write down all analytical formulas of $i\mathcal{M}^{(\xi)}_{(m)}$ ($m=\overline{1,10}$) corresponding to 10 diagrams presented in Figure 1, which provide one-loop contributions to the LFVh decay amplitude. Then the total result $i\mathcal{M}^{(\xi)}=\sum_{m=1}^{10} i\mathcal{M}^{(\xi)}_{(m)}$ is identified with $i\overline{u_a}\left(\Delta_L^{(ab)}P_L+\Delta_R^{(ab)}P_R\right)v_b$. The LFVh partial decay width is $\Gamma(h\to e_a e_b)\equiv\Gamma(h\to e_a^- e_b^+)+\Gamma(h\to e_a^+ e_b^-)\simeq m_h\left(|\Delta_L^{(ab)}|^2+|\Delta_R^{(ab)}|^2\right)/(8\pi)$, where $m_h\gg m_a, m_b$ and m_a, m_b being masses of charged leptons, respectively. The on-shell conditions for external momenta are $p_1^2=m_a^2$ ($a=1,2$), $p_2^2=m_b^2$ ($a<b=2,3$), and $q^2\equiv(p_1+p_2)^2=m_h^2$, with $m_h=125$ GeV. The internal momentum corresponding to three propagators is denoted as k, $k_1=k-p_1$, and $k_2=k+p_2$, as presented in one-loop three- and two-point diagrams (1), (7), and (8) of Figure 1. In the following, the one-loop contribution to $\Delta_{L,R}^{(ab)}$ will be calculated in the general gauge R_ξ, with every analytic formula divided into two parts: the unitary part and the remaining part dependent on the gauge parameter ξ. Then we will prove analytically that the sum of all ξ-dependent parts will vanish. We will use the transformation $\int\frac{d^4k}{(2\pi)^4}\to\int Dk$ for the dimensional regularization technique [14] to evaluate all Feynman integrals, then express the final result in terms of the Passarino-Veltman (PV) functions [15], [16], which are useful for numerical calculations with LoopTools [17]. The Feynman rules in Table 1 give the following specific formulas for one-loop contributions corresponding to the 10 diagrams depicted in Figure 1.

$$i\mathcal{M}^{(\xi)}_{(1)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\left(\frac{ig}{\sqrt{2}}U^{\nu*}_{ai}\gamma_\mu P_L\right)\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}\left(\frac{ig}{\sqrt{2}}U^{\nu}_{bi}\gamma_\nu P_L\right)v_b$$
$$\times(igm_W g_{\alpha\beta})P_W^{(\xi)\mu\alpha}(k_1^2)P_W^{(\xi)\nu\beta}(k_2^2),$$

$$i\mathcal{M}^{(\xi)}_{(2)}=\sum_{i,j=1}^{9}\int Dk\times\overline{u_a}\left(\frac{ig}{\sqrt{2}}U^{\nu*}_{ai}\gamma_\mu P_L\right)\frac{i\left(-\not{k}_1+m_{n_i}\right)}{k_1^2-m_{n_i}^2}\times\frac{-ig}{2m_W}\left(\lambda_{ij}P_L+\lambda^*_{ij}P_R\right)$$
$$\times\frac{i\left(-\not{k}_2+m_{n_j}\right)}{k_2^2-m_{n_j}^2}\times\left(\frac{ig}{\sqrt{2}}U^{\nu}_{bj}\gamma_\nu P_L\right)v_b\times P_W^{(\xi)\mu\nu}(k^2),$$

$$i\mathcal{M}^{(\xi)}_{(3)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\left(\frac{ig}{\sqrt{2}}U^{\nu*}_{ai}\gamma_\mu P_L\right)\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}\left(-\frac{ig}{2m_W}U^{\nu}_{bi}\right)(m_bP_R-m_{n_i}P_L)v_b$$
$$\times\frac{ig}{\sqrt{2}}(q+k_2)_\nu P_W^{(\xi)\mu\nu}(k_1^2)P_G^{(\xi)}(k_2^2),$$

$$i\mathcal{M}^{(\xi)}_{(4)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\left(-\frac{ig}{2m_W}U^{\nu*}_{ai}\right)(m_aP_L-m_{n_i}P_R)\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}\left(\frac{ig}{\sqrt{2}}U^{\nu}_{bi}\gamma_\nu P_L\right)v_b$$
$$\times\frac{ig}{\sqrt{2}}(k_1-q)_\mu P_G^{(\xi)}(k_1^2)P_W^{(\xi)\mu\nu}(k_2^2),$$

$$i\mathcal{M}_{(5)}^{(\xi)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\left(-\frac{ig}{\sqrt{2}m_W}U_{ai}^{\nu*}\right)(m_aP_L-m_{n_i}P_R)\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}$$

$$\times\left(-\frac{ig}{\sqrt{2}m_W}U_{bi}^{\nu}\right)(m_bP_R-m_{n_i}P_L)v_b\times\left[-\frac{igm_h^2}{2m_W}\right]P_G^{(\xi)}(k_1^2)P_G^{(\xi)}(k_2^2),$$

$$i\mathcal{M}_{(6)}^{(\xi)}=\sum_{i,j=1}^{9}\int Dk\times\overline{u_a}\left(-\frac{ig}{\sqrt{2}m_W}\right)^2U_{ai}^{\nu*}U_{bj}^{\nu}(m_aP_L-m_{n_i}P_R)\frac{i\left(-\not{k}_1+m_{n_i}\right)}{k_1^2-m_{n_i}^2}$$

$$\times\frac{-ig}{2m_W}\left(\lambda_{ij}P_L+\lambda_{ij}^*P_R\right)\frac{i\left(-\not{k}_2+m_{n_j}\right)}{k_2^2-m_{n_j}^2}(m_bP_R-m_{n_j}P_L)v_b\times P_G^{(\xi)}(k^2),$$

$$i\mathcal{M}_{(7)}^{(\xi)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\left(\frac{ig}{\sqrt{2}}U_{ai}^{\nu*}\gamma_\mu P_L\right)\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}\left(\frac{ig}{\sqrt{2}}U_{bi}^{\nu}\gamma_\nu P_L\right)$$

$$\times\frac{i(\not{p}_1+m_b)}{p_1^2-m_b^2}\times\frac{-igm_b}{2m_W}v_b\times P_W^{(\xi)\mu\nu}(k_1^2),$$

$$i\mathcal{M}_{(8)}^{(\xi)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\times\frac{-igm_a}{2m_W}\times\frac{i(-\not{p}_2+m_a)}{p_2^2-m_a^2}\left(\frac{ig}{\sqrt{2}}U_{ai}^{\nu*}\gamma_\mu P_L\right)$$

$$\times\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}\left(\frac{ig}{\sqrt{2}}U_{bi}^{\nu}\gamma_\nu P_L\right)v_b\times P_W^{(\xi)\mu\nu}(k_2^2),$$

$$i\mathcal{M}_{(9)}^{(\xi)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\left(-\frac{ig}{\sqrt{2}m_W}U_{ai}^{\nu*}\right)(m_aP_L-m_{n_i}P_R)\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}$$

$$\times\left(-\frac{ig}{\sqrt{2}m_W}U_{bi}^{\nu}\right)(m_bP_R-m_{n_i}P_L)\frac{i(\not{p}_1+m_b)}{p_1^2-m_b^2}\times\frac{-igm_b}{2m_W}v_b\times P_G^{(\xi)}(k_1^2),$$

$$i\mathcal{M}_{(10)}^{(\xi)}=\sum_{i=1}^{9}\int Dk\times\overline{u_a}\times\frac{-igm_a}{2m_W}\times\frac{i(-\not{p}_2+m_a)}{p_2^2-m_a^2}\left(-\frac{ig}{\sqrt{2}m_W}U_{ai}^{\nu*}\right)(m_aP_L-m_{n_i}P_R)$$

$$\times\frac{i\left(\not{k}+m_{n_i}\right)}{k^2-m_{n_i}^2}\left(-\frac{ig}{\sqrt{2}m_W}U_{bi}^{\nu}\right)(m_bP_R-m_{n_i}P_L)v_b\times P_G^{(\xi)}(k_2^2),\quad (6)$$

where $D_{0,X}\equiv k^2-m_X^2$ and $D_{m,X}\equiv k_m^2-m_X^2$, with $m=1,2$; and $X=n_i,G,W$. The unitary gauge relates to only four diagrams: (1), (2), (7), and (8). We will prove that the sum of all deviations between two calculations in the general gauge R_ξ and unitary is zero. Namely, the deviation from the diagrams (1) is shown as follows

$$i\delta\mathcal{M}_1\equiv i\mathcal{M}_{(1)}^{(\xi)}-i\mathcal{M}_{(1)}^{(u)}$$

$$= \frac{g^3 m_W}{2} \sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} \int Dk \frac{\overline{u_a} \gamma_\mu P_L \left(\not{k} + m_{n_i} \right) \gamma_\nu P_L v_b}{D_{0,n_i}}$$

$$\times g_{\alpha\beta} \left[P_W^{(\xi)\mu\alpha}(k_1^2) P_W^{(\xi)\nu\beta}(k_2^2) - P_W^{(u)\mu\alpha}(k_1^2) P_W^{(u)\nu\beta}(k_2^2) \right]$$

$$= \frac{g^3 m_W}{2} \sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} \int Dk \times \frac{\overline{u}_a \gamma_\mu \not{k} \gamma_\nu P_L v_b \times g_{\alpha\beta}}{D_{0,n_i}}$$

$$\times \left\{ P_W^{(T)\mu\alpha}(k_1^2) P_W^{(T)\nu\beta}(k_2^2) + P_W^{(T)\mu\alpha}(k_1^2) P_W^{(u)\nu\beta}(k_2^2) + P_W^{(u)\mu\alpha}(k_1^2) P_W^{(T)\nu\beta}(k_2^2) \right\}, \tag{7}$$

where $P_W^{(\xi)\mu\alpha}(k_1^2)$ and $P_W^{(\xi)\nu\beta}(k_2^2)$ are written in terms of the formula given in Eq. (5). After contracting all Lorentz indices, we obtain

$$i\delta\mathcal{M}_1 = -\frac{g^3}{2m_W^3} \sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} \int \frac{Dk}{D_{0,n_i} D_{1,G} D_{2,G}} \times \left[\overline{u}_a \not{k}_1 \not{k} \not{k}_2 P_L v_b \right] (k_1.k_2)$$

$$- \frac{g^3}{2m_W} \sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} \int \frac{Dk}{D_{0,n_i} D_{1,G} D_{2,W}} \times \overline{u}_a \left[\not{k}_1 \not{k} \not{k}_1 - \frac{\not{k}_1 \not{k} \not{k}_2 (k_1.k_2)}{m_W^2} \right] P_L v_b$$

$$- \frac{g^3}{2m_W} \sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} \int \frac{Dk}{D_{0,n_i} D_{1,W} D_{2,G}} \times \overline{u}_a \left[\not{k}_2 \not{k} \not{k}_2 - \frac{\not{k}_1 \not{k} \not{k}_2 (k_1.k_2)}{m_W^2} \right] P_L v_b. \tag{8}$$

Two diagrams (3) and (4) appearing only in the gauge R_ξ are expressed as follows

$$i\mathcal{M}_{(3)}^{(\xi)} = \frac{-i^5 g^3}{4m_W} \sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} \int \frac{Dk}{D_{0,n_i} D_{2,G}} \left[\overline{u_a} \gamma_\mu (m_b \not{k} P_R - m_{n_i}^2 P_L) v_b \right] \times (q + k_2)_\nu P_W^{(\xi)\mu\nu}(k_1^2)$$

$$= -\frac{g^3}{4m_W} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,W} D_{2,G}} \overline{u}_a \left[(\not{q} + \not{k}_2) - \frac{\not{k}_1 \left[k_1.(q + k_2) \right]}{m_W^2} \right] (-\not{k} \not{p}_2 - m_{n_i}^2) P_L v_b$$

$$- \frac{g^3}{4m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,G} D_{2,G}} \left[\overline{u}_a \not{k}_1 (-\not{k} \not{p}_2 - m_{n_i}^2) P_L v_b \right] \times \left[k_1.(q + k_2) \right], \tag{9}$$

and

$$i\mathcal{M}_{(4)}^{(\xi)} = -\frac{g^3}{4m_W} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,G} D_{2,W}} \overline{u}_a (\not{p}_1 \not{k} - m_{n_i}^2) \left[(\not{k}_1 - \not{q}) - \frac{\not{k}_2 \left[k_2.(k_1 - q) \right]}{m_W^2} \right] P_L v_b$$

$$- \frac{g^3}{4m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,G} D_{2,G}} \overline{u}_a (\not{p}_1 \not{k} - m_{n_i}^2) \not{k}_2 \left[k_2.(k_1 - q) \right] P_L v_b. \tag{10}$$

In the same way as the above calculation, we get the following deviations after subtracting the unitary-gauge parts

$$i\delta\mathcal{M}_{(2)} \equiv i\mathcal{M}_{(2)}^{(\xi)} - i\mathcal{M}_{(2)}^{(u)} = \frac{g^3}{4} \sum_{i,j=1}^{9} U_{ai}^{\nu *} U_{bj}^{\nu} \int Dk \times \frac{\overline{u}_a \left(\lambda_{ij}^* m_{n_j} \not{k} \not{k}_1 \not{k} P_L + \lambda_{ij} m_{n_i} \not{k} \not{k}_2 \not{k} P_L \right) v_b}{m_W^3 D_{0,G} D_{1,n_i} D_{2,n_j}},$$

$$i\delta\mathcal{M}_{(7)} = i\mathcal{M}_{(7)}^{(\xi)} - i\mathcal{M}_{(7)}^{(u)} = \frac{-g^3 m_b}{4(m_a^2 - m_b^2)} \sum_{i=1}^{9} U_{ai}^{\nu*} U_{bi}^{\nu} \int Dk \times \frac{\overline{u_a} \not{k}_1 \not{k} \not{k}_1 P_L (\not{p}_1 + m_b) v_b}{m_W^3 D_{0,n_i} D_{1,G}},$$

$$i\delta\mathcal{M}_{(8)} = i\mathcal{M}_{(8)}^{(\xi)} - i\mathcal{M}_{(8)}^{(u)} = \frac{-g^3 m_a}{4(m_b^2 - m_a^2)} \sum_{i=1}^{9} U_{ai}^{\nu*} U_{bi}^{\nu} \int Dk \times \frac{\overline{u_a}(-\not{p}_2 + m_a) \not{k}_2 \not{k} \not{k}_2 P_L v_b}{m_W^3 D_{0,n_i} D_{2,G}}. \quad (11)$$

The formulas for all remaining diagrams not appearing in the unitary gauge are

$$i\mathcal{M}_{(5)}^{(\xi)} = \frac{-i^6 g^3 m_h^2}{4 m_W^3} \sum_{i=1}^{9} U_{ai}^{\nu*} U_{bi}^{\nu} \int Dk \times \frac{\overline{u_a}(m_a P_L - m_{n_i} P_R)\left(\not{k} + m_{n_i}\right)(m_b P_R - m_{n_i} P_L) v_b}{D_{0,n_i} D_{1,G} D_{2,G}}$$

$$= \frac{g^3 m_h^2}{4 m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu*} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,G} D_{2,G}} \times \overline{u_a}\left[-\left(m_a \not{k} - m_{n_i}^2\right)\not{p}_2 + m_{n_i}^2 \not{k}_1\right] P_L v_b,$$

$$i\mathcal{M}_{(6)}^{(\xi)} = \frac{-g^3}{4} \sum_{i,j=1}^{9} \int \frac{U_{ai}^{\nu*} U_{bj}^{\nu} Dk}{m_W^3 D_{0,G} D_{1,n_i} D_{2,n_j}}$$

$$\times \overline{u_a}\left[\lambda_{ij}^* m_{n_j}\left(\not{p}_1 \not{k}_1 + m_{n_i}^2\right)\not{k} + \lambda_{ij} m_{n_i} \not{k}\left(-\not{k}_2 \not{p}_2 + m_{n_j}^2\right)\right] P_L v_b,$$

$$i\mathcal{M}_{(9)}^{(\xi)} = \frac{g^3 m_b}{4(m_a^2 - m_b^2)} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu*} U_{bi}^{\nu} Dk}{m_W^3 D_{0,n_i} D_{1,G}} \times \overline{u_a}\left[m_b\left(m_a \not{k} - m_{n_i}^2\right)P_R + m_{n_i}^2 \not{k}_1 P_L\right](\not{p}_1 + m_b) v_b,$$

$$i\mathcal{M}_{(10)}^{(\xi)} = \frac{g^3 m_a}{4(m_b^2 - m_a^2)} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu*} U_{bi}^{\nu} Dk}{m_W^3 D_{0,n_i} D_{2,G}} \times \overline{u_a}(-\not{p}_2 + m_a)\left[-\left(m_a \not{k} - m_{n_i}^2\right)\not{p}_2 + m_{n_i}^2 \not{k}_1\right] P_L v_b, \quad (12)$$

where the on-shell Dirac equations $(\not{k}_2 P_L + m_b P_R) v_b = (\not{k}_2 P_L + P_R(-\not{p}_2)) v_b = \not{k} P_L$ and $\overline{u_a}(\not{k}_1 P_L + m_a P_L) = \overline{u_a} \not{k} P_L$ to derive the result in the last line of $i\mathcal{M}_{(6)}^{(\xi)}$.

Summing two deviations relating to one-loop two-point diagrams (7) and (9), we get

$$i\delta\mathcal{M}_{(7+9)} = i\delta\mathcal{M}_{(7)} + i\mathcal{M}_{(9)}$$

$$= \frac{-g^3 m_b}{4(m_a^2 - m_b^2) m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu*} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,G}}$$

$$\times \overline{u_a}\left\{-\left[(D_{0,n_i} - m_a^2)\not{k}_1 - m_a k_1^2\right] P_L - m_b\left(m_a \not{k} - m_{n_i}^2\right) P_R\right\}(\not{p}_1 + m_b) v_b, \quad (13)$$

where we have used $\not{k}_1 \not{k} \not{k}_1 = \not{k}(k^2 - m_a^2) + m_a(m_a^2 - k^2 - k_1^2) = \not{k}_1(k^2 - m_a^2) - m_a k_1^2$. Next, the sum of deviations relating to two similar one-loop two-point diagrams (8), and (10) is

$$i\delta\mathcal{M}_{(8+10)} = i\delta\mathcal{M}_{(8)} + i\delta\mathcal{M}_{(10)}$$

$$= \frac{-g^3 m_a}{4(m_b^2 - m_a^2)m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{2,G}}$$

$$\times \overline{u_a}(-\not{p}_2 + m_a)\left\{ -\not{k}_2 \not{k} \not{k}_2 P_L - \left[m_a m_b \not{k} P_R + m_{n_i}^2 \not{k} P_L - m_{n_i}^2 (m_a P_L + m_b P_R) \right] \right\} v_b$$

$$= \frac{g^3 m_a}{4(m_a^2 - m_b^2)m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{2,G}}$$

$$\times \overline{u_a}(m_a - \not{p}_2)\left\{ -D_{0,n_i} \not{k}_2 + k^2 \not{p}_2 + \not{p}_2 \not{k} \not{p}_2 + m_a \not{k} \not{p}_2 + m_a m_{n_i}^2 \right\} P_L v_b, \quad (14)$$

where we have used $\not{k}_2 \not{k} \not{k}_2 = k^2(\not{k}_2 + \not{p}_2) + \not{p}_2 \not{k} \not{p}_2$.

From now on, we will use the unitary property of U^{ν} that $\sum_{i=1}^{9} U_{ai}^{\nu *} U_{bi}^{\nu} = \delta_{ab} = 0$, with $a \neq b$, for terms being independent of the index i, namely

$$\sum_{i=1}^{9} \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk \times D_{0,n_i} \not{k}_m}{D_{0,n_i} D_{m,G}} = \frac{Dk \times \not{k}_m \times \delta_{ab}}{D_{m,G}} = 0, \quad (15)$$

with $m = 1, 2$ and $a \neq b$. This leads to beautiful and simple formulas of the mentioned deviations as follows

$$i\delta\mathcal{M}_{(7+9)} = \frac{g^3 m_b}{4m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{1,G}} \times \overline{u_a} \left[\left(-m_a \not{k} + m_{n_i}^2 \right) P_R \right] v_b,$$

$$i\delta\mathcal{M}_{(8+10)} = \frac{g^3 m_a}{4m_W^3} \sum_{i=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bi}^{\nu} Dk}{D_{0,n_i} D_{2,G}} \times \overline{u_a} \left[\left(\not{k} \not{p}_2 + m_{n_i}^2 \right) P_L \right] v_b. \quad (16)$$

Regarding the sum of the two diagrams (2) and (6) containing two neutrino propagators, we have

$$i\delta\mathcal{M}_{(2+6)} = -\frac{g^3}{4m_W^3} \sum_{i,j=1}^{9} \int \frac{U_{ai}^{\nu *} U_{bj}^{\nu} Dk}{D_{0,G} D_{1,n_i} D_{2,n_j}}$$

$$\times \overline{u_a} \left\{ \lambda_{ij}^* m_{n_j} \left[\left(m_a \not{k}_1 + m_{n_i}^2 \right) \not{k} - \not{k} \not{k}_1 \not{k} \right] P_L - \right.$$

$$\left. + \lambda_{ij} m_{n_i} \left[\not{k} \left(m_b \not{k}_2 P_R + m_{n_j}^2 P_L \right) - \not{k} \not{k}_2 \not{k} P_L \right] \right\} v_b$$

$$= \frac{g^3}{4m_W^3} \sum_{i,j=1}^{9} \int U_{ai}^{\nu *} U_{bj}^{\nu} Dk \times \overline{u_a} \left(\frac{\lambda_{ij}^* m_{n_j}}{D_{0,G} D_{2,n_j}} + \frac{\lambda_{ij} m_{n_i}}{D_{0,G} D_{1,n_i}} \right) \not{k} P_L v_b$$

$$= \frac{g^3}{4m_W^3} \sum_{i}^{9} \int U_{ai}^{\nu *} U_{bi}^{\nu} Dk \times \overline{u_a} \left(\frac{m_{n_i}^2}{D_{0,G} D_{2,n_i}} + \frac{m_{n_i}^2}{D_{0,G} D_{1,n_i}} \right) \not{k} P_L v_b, \quad (17)$$

where we have used the following property of λ_{ij} given in Eq. (1)

$$\sum_{i=1}^{9} U^{\nu*}_{ai}\lambda^*_{ij} = \sum_{d=1}^{3}\left[\left(\mathcal{M}^{\nu*}\right)_{ad} + U^{\nu*}_{dj} m_{n_j}\delta_{ad}\right] = U^{\nu*}_{aj} m_{n_j},\ \sum_{j=1}^{9} U^{\nu}_{bj}\lambda_{ij} = U^{\nu}_{bi} m_{n_i}.$$

3. Results and discussion

Now we focus on the sum of the deviations relating to four one-loop three-point diagrams containing two boson propagators, using well-known properties that $q^2 = m_h^2$ and $q = k_2 - k_1$, which result in useful relations for our calculation, such as $k_1.(q+k_2) = -k_1^2 + 2k_1.k_2 = k_2^2 - m_h^2$ and $k_2.(k_1 - q) = k_1^2 - m_h^2 = 2k_1.k_2 - k_2^2$. Defining $i\delta\mathcal{M}^{(\xi)}_{nBB} = i\delta\mathcal{M}^{(\xi)}_1 + i\delta\mathcal{M}^{(\xi)}_3 + i\delta\mathcal{M}^{(\xi)}_4 + i\delta\mathcal{M}^{(\xi)}_5$, we separate $i\delta\mathcal{M}^{(\xi)}_{nBB}$ into three following parts

$$i\delta\mathcal{M}^{(\xi)}_{nBB} = -\frac{g^3}{4m_W^3}\sum_{i=1}^{9} U^{\nu*}_{ai}U^{\nu}_{bi}\int \frac{Dk}{D_{0,n_i}} \times \overline{u_a}\left[\frac{F_{iGG}}{D_{1,G}D_{2,G}} + \frac{F_{iGW}}{D_{1,G}D_{2,W}} + \frac{F_{iWG}}{D_{1,W}D_{2,G}}\right]P_L v_b,$$

where

$$\begin{aligned}
F_{iGG} &= \left(\not{k}_1\not{k}\not{k}_2\right)(k_1^2 + k_2^2 - m_h^2) + \not{k}_1(-\not{k}\not{p}_2 - m_{n_i}^2)\times\left(k_2^2 - m_h^2\right) \\
&\quad + (m_a\not{k} - m_{n_i}^2)\not{k}_2\left(k_1^2 - m_h^2\right) - m_h^2\left[-m_a\not{k}\not{p}_2 + m_{n_i}^2\not{k} - m_{n_i}^2(m_a - \not{p}_2)\right], \\
F_{iGW} &= 2m_W^2\not{k}_1\not{k}\not{k}_1 - 2\not{k}_1\not{k}\not{k}_2(k_1.k_2) + (m_a\not{k} - m_{n_i}^2)\left[m_W^2(\not{k}_1 - \not{q}) - \not{k}_2\left(2k_1.k_2 - k_2^2\right)\right], \\
F_{iWG} &= 2m_W^2\not{k}_2\not{k}\not{k}_2 - 2\not{k}_1\not{k}\not{k}_2(k_1.k_2) + \left[m_W^2(\not{q} + \not{k}_2) - \not{k}_1\left(2k_1.k_2 + k_1^2\right)\right](-\not{k}\not{p}_2 - m_{n_i}^2).
\end{aligned} \quad (18)$$

After some simple intermediate calculation, we derive that

$$\begin{aligned}
F_{iGG} &= D_{0,n_i}\left(k_1^2\not{k}_2 + k_2^2\not{k}_1 - m_h^2\not{k}\right), \\
F_{iGW} &= 2D_{0,n_i}\left[m_W^2\not{k}_1 - 2(k_1.k_2)\not{k}_2\right] + (m_a\not{k} - m_{n_i}^2)\not{k}_2 D_{2,W}, \\
F_{iWG} &= 2D_{0,n_i}\left[m_W^2\not{k}_2 - 2(k_1.k_2)\not{k}_1\right] - \not{k}_1(\not{k}\not{p}_2 + m_{n_i}^2)D_{1,W}.
\end{aligned} \quad (19)$$

We emphasize that the simple form of F_{iGG} arises from the model-dependent couplings $hG^+_W G^-_W$ given in Table 1. Also, F_{iGW} and F_{iWW} result from the model-dependent of the couplings $hW^{\pm}G^{\mu}_W$. Consequently, all terms proportional to D_{0,n_i} appearing in Eq. (19) vanish when inserting in $i\delta\mathcal{M}^{(\xi)}_{nBB}$ then applying the unitary property of U^{ν}. Now, the final sum of all deviations is written in terms of the PV-functions as follows

$$\begin{aligned}
i\delta\mathcal{M} &= i\delta\mathcal{M}^{(\xi)}_{nBB} + i\delta\mathcal{M}_{(2+6)} + i\delta\mathcal{M}_{(7+9)} + i\delta\mathcal{M}_{(8+10)} \\
&= \frac{g^3}{4m_W^3}\sum_{i=1}^{9} U^{\nu*}_{ai}U^{\nu}_{bi}\int Dk \times \overline{u_a}\left[\frac{(-m_a\not{k} + m_{n_i}^2)\not{k}}{D_{0,n_i}D_{1,G}} + \frac{m_{n_i}^2\not{k}}{D_{0,G}D_{1,n_i}}\right]P_L v_b
\end{aligned}$$

$$+\frac{g^3}{4m_W^3}\sum_{i=1}^{9}U_{ai}^{\nu*}U_{bi}^{\nu}\int Dk\times\overline{u_a}\left[\frac{\not{k}(\not{k}\not{p}_2+m_{n_i}^2)}{D_{0,n_i}D_{2,G}}+\frac{m_{n_i}^2\not{k}}{D_{0,G}D_{2,n_i}}\right]P_L v_b$$

$$\propto -m_a\sum_{i=1}^{9}U_{ai}^{\nu*}U_{bi}^{\nu}\left\{-A_0(m_{G_W})+m_{n_i}^2\left[B_1(p_a^2;m_{G_W}^2,m_{n_i}^2)+\left(B_1+B_0\right)(p_a^2;m_{n_i}^2,m_{G_W}^2)\right]\right\}$$

$$+\sum_{i=1}^{9}U_{ai}^{\nu*}U_{bi}^{\nu}\left\{-A_0(m_{G_W})+m_{n_i}^2\left[B_1(p_a^2;m_{G_W}^2,m_{n_i}^2)+\left(B_1+B_0\right)(p_a^2;m_{n_i}^2,m_{G_W}^2)\right]\right\}\not{p}_2, \tag{20}$$

where we use the standard notations for PV-funtions given in Refs. [18], [19], consistent with Ref. [9] after redefining opposite signs in PV-function B_1 and C_1. The final result expressed by the last formulas given in Eq. (20) is exactly zero because of the two following points: i) $B_1(p_a^2;m_{G_W}^2,m_{n_i}^2)+B_1(p_a^2;m_{n_i}^2,m_{G_W}^2)+B_0(p_a^2;m_{n_i}^2,m_{G_W}^2)=0$; ii) and the unity that U^{ν} gives. $\sum_{i=1}^{9}U_{ai}^{\nu*}U_{bi}^{\nu}A_0(m_{G_W})\propto\delta_{ab}=0$. We confirm that our calculation can be cross-checked step by step using the FORM package [20]. Now we finish our calculation with a final conclusion that the formula $i\mathcal{M}^{\xi}$ for the sum of all one-loop contributions to the LFVh amplitude is completely independent of the unphysical parameter ξ of the general gauge R_{ξ}.

4. Conclusions

In this work, using standard Feynman rules and mathematical methods, we prove precisely that the total one-loop contributions to the LFVh amplitude of the GSS extension of the SM are independent of the gauge parameter ξ. This work again confirms the general conclusion that the final physical amplitude is always independent of an unphysical parameter like ξ. However, the calculations require precise forms of the gauge-dependent couplings involving the Goldstone boson $G_W^{\pm}$. Finally, we would like to note here a very interesting feature we derive from our calculation in the general gauge R_{ξ}. Namely, the sum of the class of similar Feynman diagrams, which are the same if the propagators of gauge and Goldstone bosons are distinguishable, do not vanish as expected; see the formulas of $i\delta\mathcal{M}_{nBB}^{(\xi)}$, $i\delta\mathcal{M}_{(2+6)}^{(\xi)}$, $i\delta\mathcal{M}_{(7+9)}^{(\xi)}$, and $i\delta\mathcal{M}_{(8+10)}^{(\xi)}$. However, these sums are all written in terms of the two-point PV functions so that one can easily use the well-known relations among these functions to derive the final result. This property may be useful for further studies of the gauge dependence properties of LFVh amplitudes in BSM theories with many new gauge bosons apart from $W^{\pm}$.

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