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Some applications of directional subdifferentials at infinity to vector optimization problems

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Abstract

This paper presents a comprehensive investigation into several advanced applications of variational analysis, with a particular focus on the roles of special normal cones and directional subdifferentials at infinity. By extending classical tools of nonsmooth and variational analysis to unbounded settings, we establish directional optimality conditions at infinity and derive sufficient conditions ensuring the existence and compactness of global solution sets for vector optimization problems. The theoretical results obtained contribute to a deeper understanding of optimization behavior in asymptotic regimes. Furthermore, illustrative examples are provided to demonstrate the validity and applicability of the proposed results, highlighting their potential use in future research on infinite-dimensional and unbounded optimization problems.

Keywords: vector optimization problem, optimality conditions, existence, directional normal cones at infinity, directional subdifferentials at infinity

1. Introduction

The Mordukhovich subdifferential constitutes a fundamental instrument in modern variational analysis, offering a rigorous framework for investigating nonsmooth and nonconvex functions. It extends classical subdifferential constructions and plays a pivotal role in deriving optimality conditions and analyzing stability phenomena in optimization and control theory (see, e.g., [1]–[8], [9]–[13]).

In recent developments, Kim et al. [14] introduced the notions of normal cones at infinity for unbounded sets, together with limiting and singular subdifferentials at infinity for extended real-valued functions. Their framework enables a systematic description of function behavior “at infinity,” together

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with calculus rules and characterizations of Lipschitz continuity in that regime. These constructions have proved useful in various optimization contexts, including the analysis of weak sharp minima, necessary optimality conditions, and stability properties. Subsequently, Tung and Son [15] investigated Clarke-type tangent cones at infinity and developed a corresponding subdifferential calculus to establish necessary optimality conditions. Building upon these results, Tuyen [16] derived sufficient conditions for the existence of error bounds at infinity in lower semicontinuous inequality systems and formulated optimality conditions for constrained problems. Moreover, Tuyen et al. [17] extended these ideas to the setting of nonsmooth minimax optimization.

More recently, Anh and Hung [18] studied normal cones and subdifferentials relative to a set, providing refined calculus rules and extending earlier results from [19]–[21]. They further introduced notions of normal cones and subdifferentials with respect to a set at infinity and applied these concepts to derive optimality conditions, investigate the compactness of solution sets, and verify coercivity properties for unbounded optimization problems. In [22], the authors explored normal cones and subdifferentials at infinity in a prescribed direction and demonstrated their applicability to nonsmooth optimization problems.

Motivated by these recent advances, the present paper develops several applications to vector optimization problems by employing the results of [22]. The paper is organized as follows. Section 2 reviews fundamental notions and preliminary results from variational analysis, focusing on special normal cones and subdifferentials in directions at infinity. Section 3 presents applications to vector optimization problems.

2. Preliminaries

2.1. Notation and Definition

Throughout the paper, denote $\mathbb{N} := \{1, 2, \dots\}$ and let \mathbb{R}^n be the Euclidean space with the usual scalar product $\langle \cdot, \cdot \rangle$ the corresponding Euclidean norm $\|\cdot\|$ where $n \in \mathbb{N}$. The closed unit ball and the nonnegative orthant in \mathbb{R}^n are denoted, respectively, by \mathbb{B} and \mathbb{R}_+^n . The closed ball centered at the origin with radius $R > 0$ is denoted by \mathbb{B}_R . Let D be a subset of \mathbb{R}^n . We say that D is *locally closed* if for any $x \in D$ there is a neighborhood U of x such that $D \cap U$ is closed. The interior, the boundary, the positive hull and the convex hull of D are denoted, respectively, by $\text{int}D$, $\text{bd}D$, $\text{pos}D$ and $\text{co}D$. As usual, $\mathbb{S} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is the unit sphere in \mathbb{R}^n , $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is the extended real line. The notation $x \rightarrow \infty$ means that $\|x\| \rightarrow \infty$. The *asymptotic cone* of D , denoted by D^∞ , is defined by

$$D^\infty := \left\{ u \in \mathbb{R}^n : \exists t_k \rightarrow +\infty, x_k \in D, \frac{x_k}{t_k} \rightarrow u \right\}.$$

When $D = \emptyset$, we put $\emptyset^\infty := \emptyset$. It follows from [3, Proposition 2.1.2] that D is *bounded* if and only if $D^\infty = \{0\}$.

Given an extended real-valued function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$. The *effective domain* and the *epigraph* of f are denoted, respectively, by

$$\text{dom}f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$$

and

$$\text{epif} := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

We say that f is *proper* (resp., *proper at infinity*) if $\text{dom}f$ is nonempty (resp., $\text{dom}f$ is unbounded). The function f is called *lower semicontinuous* (l.s.c.) if its epigraph is closed.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The *domain* of F are given by

$$\text{dom}F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}.$$

The set-valued mapping F is called *proper* if $\text{dom}F \neq \emptyset$.

For a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the Painlevé-Kuratowski outer/upper limit of F at \bar{x} is defined by

$$\text{Limsup}_{x \rightarrow \bar{x}} F(x) := \{y \in \mathbb{R}^m \mid \exists x_k \rightarrow \bar{x}, y_k \rightarrow y \text{ with } y_k \in F(x_k) \quad \forall k \in \mathbb{N}\}$$

and as $x \rightarrow \infty$ we define

$$\text{Limsup}_{x \rightarrow \infty} F(x) := \{y \in \mathbb{R}^m \mid \exists x_k \rightarrow \infty, y_k \rightarrow y \text{ with } y_k \in F(x_k) \quad \forall k \in \mathbb{N}\}.$$

2.2. Normal Cones and Subdifferentials

Definition 2.1. Let Ω be a nonempty subset of \mathbb{R}^n and let $\bar{x} \in \Omega$ (see [10], [11]).

(i) The regular/Fréchet normal cone to Ω at \bar{x} is defined by

$$\widehat{N}(\bar{x}; \Omega) = \{v \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \rightarrow \bar{x}$ and $x \in \Omega$.

(ii) The limiting/Mordukhovich normal cone to Ω at \bar{x} is given by

$$N(\bar{x}; \Omega) = \text{Limsup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x; \Omega).$$

When $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) := \emptyset$.

Definition 2.2. Given a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and a point $\bar{x} \in \text{dom}f$.

(i) The regular/Fréchet subdifferential of f at \bar{x} is defined by

$$\widehat{\partial}f(\bar{x}) := \{v \in \mathbb{R}^n \mid (v, -1) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epif})\}.$$

(ii) The limiting/Mordukhovich subdifferential and the limiting/Mordukhovich singular subdifferential of f at \bar{x} are defined, respectively, by

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^n \mid (v, -1) \in N((\bar{x}, f(\bar{x})); \text{epif})\},$$

and

$$\partial^\infty f(\bar{x}) := \{v \in \mathbb{R}^n \mid (v, 0) \in N((\bar{x}, f(\bar{x})); \text{epif})\}.$$

2.3. Normal Cones and Subdifferentials at Infinity

This subsection provides definitions of normal cones and subdifferentials at infinity (see [14]).

Definition 2.3. Let Ω be an unbounded subset in \mathbb{R}^n . The *normal cone to the set Ω at infinity* is defined by

$$N(\infty; \Omega) := \text{Limsup}_{x \xrightarrow{\Omega} \infty} \widehat{N}(x; \Omega),$$

where $x \xrightarrow{\Omega} \infty$ means that $\|x\| \rightarrow \infty$ and $x \in \Omega$.

Definition 2.4. Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an l.s.c. function that is proper at infinity. The *limiting/Mordukhovich and the singular subdifferentials* of f at infinity are defined, respectively, by

$$\partial f(\infty) := \{u \in \mathbb{R}^n \mid (u, -1) \in \mathcal{N}\},$$

$$\partial^\infty f(\infty) := \{u \in \mathbb{R}^n \mid (u, 0) \in \mathcal{N}\},$$

where $\mathcal{N} := \text{Limsup}_{x \rightarrow \infty} N((x, f(x)); \text{epif})$.

Definition 2.5. (see [14], [15]) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an l.s.c. function. We say that f is *Lipschitz at infinity* if there exist constants $L > 0$ and $R > 0$ such that

$$|f(x) - f(x')| \leq L\|x - x'\| \quad \text{for all } x, x' \in \mathbb{R}^n \setminus \mathbb{B}_R.$$

2.4. Directional Normal Cones and Subdifferentials at Infinity

In this subsection, we introduce directional normal cones at infinity for unbounded sets, along with directional limiting and singular subdifferentials at infinity for extended real-valued functions (see [22]).

Let Ω be a locally closed and unbounded subset in \mathbb{R}^n and $u \in \mathbb{S}$.

Definition 2.6. The normal cone to Ω in direction u at infinity, denoted by $N_\Omega(\infty; u)$, is defined by

$$N_\Omega(\infty; u) := \text{Limsup}_{x \xrightarrow{\Omega, u} \infty} \widehat{N}(x; \Omega),$$

i.e.,

$$N_\Omega(\infty; u) := \left\{ \xi \in \mathbb{R}^n \mid \exists x_k \xrightarrow{\Omega} \infty, \frac{x_k}{\|x_k\|} \rightarrow u, \xi_k \in \widehat{N}(x_k; \Omega), \xi_k \rightarrow \xi \right\}.$$

It is easy to show that if $u \notin \Omega^\infty$, then $N_\Omega(\infty; u) = \emptyset$. So, we consider the case $u \in \Omega^\infty \cap \mathbb{S}$.

Definition 2.7. Let $u \in \mathbb{R}^n$. A neighborhood of the infinity in direction u is defined by

$$V_{R, \delta}(\infty; u) := \{z \in \mathbb{R}^n \setminus \mathbb{B}_R : \|z\| \|u\| - u \|z\| \leq \delta \|z\| \|u\|\}$$

for some $R > 0$ and $\delta > 0$.

By directly computing, we have

$$V_{R, \delta}(\infty; u) = \begin{cases} \mathbb{R}^n \setminus \mathbb{B}_R, & \text{if } u = 0, \\ \left\{ z \in \mathbb{R}^n \setminus \mathbb{B}_R : \left\| \frac{z}{\|z\|} - \frac{u}{\|u\|} \right\| \leq \delta \right\}, & \text{if } u \neq 0. \end{cases}$$

Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an l.s.c. function and $u \in \mathbb{S}$ and we assume that $\text{dom} f$ is unbounded in direction u , i.e., $u \in (\text{dom} f)^\infty$.

Definition 2.8. The limiting and the singular subdifferentials of f in direction u at infinity are defined, respectively, by

$$\begin{aligned} \partial f(\infty; u) &:= \left\{ \xi \in \mathbb{R}^n \mid (\xi, -1) \in \text{Limsup}_{x \xrightarrow{\text{epif}, u} \infty, r \geq f(x)} \widehat{N}((x, r); \text{epif}) \right\}, \\ \partial^\infty f(\infty; u) &:= \left\{ \xi \in \mathbb{R}^n \mid (\xi, 0) \in \text{Limsup}_{x \xrightarrow{\text{epif}, u} \infty, r \geq f(x)} \widehat{N}((x, r); \text{epif}) \right\}. \end{aligned}$$

Definition 2.9. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued function and $u \in \mathbb{R}^n$. We say that f is *Lipschitz at infinity in direction u* if there exist $L > 0$, $R > 0$, and $\delta > 0$ such that

$$|f(x) - f(y)| \leq L\|x - y\| \quad \forall x, y \in V_{R, \delta}(\infty; u).$$

Proposition 2.1. (see [22, Proposition 3.13]) The function f is Lipschitz at infinity in direction u if and only if $\partial^\infty f(\infty; u) = \{0\}$. In this case, the set $\partial f(\infty; u)$ is nonempty and compact.

Proposition 2.2. (see [22, Proposition 3.15]) Let Ω be an unbounded subset in \mathbb{R}^n and $u \in \Omega^\infty \cap \mathbb{S}$. Consider the following set

$$S := \{x \in \Omega \mid g_i(x) \leq 0, i = 1, \dots, m, h_j(x) = 0, j = 1, \dots, p\},$$

where $g_i, h_j: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, i = 1, \dots, m, j = 1, \dots, p$, are Lipschitz at infinity in direction u . If S is unbounded and satisfies the limiting constraint qualification at infinity in direction u , i.e., there do not exist $\lambda_i \geq 0, i = 1, \dots, m$, and $\mu_j \geq 0, j = 1, \dots, p$, not all zero, such that

$$0 \in \sum_{i=1}^m \lambda_i \partial g_i(\infty; u) + \sum_{j=1}^p \mu_j [\partial h_j(\infty; u) \cup \partial(-h_j)(\infty; u)] + N_\Omega(\infty; u), \quad (LCQ_u^\infty)$$

then

$$N_S(\infty; u) \subset \text{pos} \left\{ \bigcup_{i=1}^m \partial g_i(\infty; u), \bigcup_{j=1}^p [\partial h_j(\infty; u) \cup \partial(-h_j)(\infty; u)] \right\} + N_\Omega(\infty; u).$$

2.5. Directional Optimality Conditions at Infinity

In this subsection, we present several applications, including directional optimality conditions at infinity, existence, and compactness of the solution set.

Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an l.s.c. function and Ω be a nonempty and closed subset of \mathbb{R}^n . We consider the following optimization problem

$$\min_{x \in \Omega} f(x), \quad (P)$$

We always assume that:

$$(C_1) \text{ dom } f \cap \Omega \text{ is unbounded.}$$

$$(C_2) f \text{ is bounded from below on } \Omega \text{ i.e., } f_* := \inf_{x \in \Omega} f(x) \text{ is finite.}$$

By [22], we have the following theorems.

Theorem 2.1. (see [22]) Let $u \in \Omega^\infty \cap \mathbb{S}$. Assume that the following condition holds:

$$\partial^\infty f(\infty; u) \cap (-N_\Omega(\infty; u)) = \{0\}. \quad (1)$$

If there exists a sequence $x_k \xrightarrow{\Omega, u} \infty$ such that $f(x_k) \rightarrow f_* := \inf_{x \in \Omega} f(x)$, then $0 \in \partial f(\infty; u) + N_\Omega(\infty; u)$.

Theorem 2.2. (see [22]) Assume that (1) and the following condition

$$0 \notin \partial f(\infty; u) + N_\Omega(\infty; u)$$

hold for all $u \in \Omega^\infty \cap \mathbb{S}$, then $\text{Sol}(P)$ is nonempty and compact.

3. Applications to vector optimization problems

Let Ω be a nonempty closed, unbounded subset of $\mathbb{R}^n, I = \{1, \dots, m\}, J = \{1, \dots, p\}$ and $f_i, g_j: \mathbb{R}^n \rightarrow \mathbb{R}$ are l.s.c, Lipschitz at infinity in direction u , for some $u \in \Omega^\infty \cap \mathbb{S}$, for all $i \in I, j \in J$.

$$\text{Denote } f := (f_1, \dots, f_m), g := (g_1, \dots, g_p), \varphi := \max_{i \in I} f_i; \mathcal{F} := \{x \in \Omega \mid g_j(x) \leq 0, j \in J\}.$$

In this section, we always assume that \mathcal{F} is nonempty and unbounded and φ is bounded from below on \mathcal{F} . For some $u \in \Omega^\infty \cap \mathbb{S}$, we consider the following constraint qualification condition in direction u at infinity:

$$\text{There do not exist } \beta_j \geq 0, j \in J \text{ such that } \sum_{j=1}^p \beta_j > 0 \text{ and} \tag{H(u)}$$

$$0 \in \sum_{j=1}^p \beta_j \partial g_j(\infty; u) + N_{\Omega}(\infty; u).$$

Let us consider a minimax programming problem

$$\min_{x \in \mathcal{F}} \max_{i \in I} f_i(x) \tag{MNP}$$

Theorem 3.1. For any $u \in \Omega^\infty \cap \mathbb{S}$, the following statements hold:

$$\partial \left(\max_{1 \leq i \leq m} f_i \right) (\infty; u) \subseteq \text{co} \{ \partial f_1(\infty; u), \dots, \partial f_m(\infty; u) \}, \tag{2}$$

$$\partial^\infty \left(\max_{1 \leq i \leq m} f_i \right) (\infty, u) = \{0\}.$$

Proof. It is easy to show that since f_1, \dots, f_m are Lipschitz at infinity in the direction u , the function $\max_{1 \leq i \leq m} f_i$ is also Lipschitz at infinity in the direction u . By Proposition 2.1, we have

$$\partial^\infty \left(\max_{1 \leq i \leq m} f_i \right) (\infty, u) = \{0\}.$$

We now prove inclusion (2). Take any $\xi \in \partial \left(\max_{1 \leq i \leq m} f_i \right) (\infty, u)$, then by [22, Proposition 3.6] there exist sequence $x_k \xrightarrow{u} \infty$ and $\xi_k \in \partial \left(\max_{1 \leq i \leq m} f_i \right) (x_k)$ such that $\xi_k \rightarrow \xi$ as $k \rightarrow \infty$.

Using an argument similar as in the proof of Proposition 2.1 and since f_1, \dots, f_m are Lipschitz at infinity in the direction u , we may assume that $f_i, i = 1, \dots, m$ are locally Lipschitz around x_k with the same constant M , for all k large enough. By [11, Theorem 1.22], $\|v\| \leq M$ for all $v \in \partial f_i(x_k), i = 1, \dots, m$ and k large enough.

By [17, Lemma 2.2], we have

$$\partial \left(\max_{1 \leq i \leq m} f_i \right) (x_k) \subseteq \text{co} \{ \partial f_1(x_k), \dots, \partial f_m(x_k) \}$$

Since $\xi_k \in \partial \left(\max_{1 \leq i \leq m} f_i \right) (x_k)$, there exist $\alpha_k^i \geq 0, i = 1, \dots, m, \sum_{i=1}^m \alpha_k^i = 1$ and $\xi_k^i \in \partial f_i(x_k), i = 1, \dots, m$ such that $\xi_k = \sum_{i=1}^m \alpha_k^i \xi_k^i$. From the boundedness of the sequence ξ_k^i and α_k^i , and by passing to a subsequence if necessary, we may assume that $\xi_k^i \rightarrow \xi^i, \alpha_k^i \rightarrow \alpha^i, i = 1, \dots, m$ as $k \rightarrow \infty$. And as $k \rightarrow \infty, x_k \xrightarrow{u} \infty$ then $\xi^i \in \partial f_i(\infty, u)$. Hence, $\xi = \sum_{i=1}^m \alpha^i \xi^i$ where $\xi^i \in \partial f_i(\infty, u), \alpha^i \geq 0, \sum_{i=1}^m \alpha^i = 1$. This mean that $\xi \in \text{co} \{ \partial f_1(\infty, u), \dots, \partial f_m(\infty, u) \}$ and (2) is proven. The proof is complete. \square

The following theorem gives a necessary optimality condition of Karush-Kuhn-Tucker (KKT) type at infinity to problem (MNP).

Theorem 3.2. Let $u \in \Omega^\infty \cap \mathbb{S}$. Assume that the condition H(u) holds. If there is a sequence $x_k \xrightarrow{\mathcal{F}, u} \infty$ such that $\varphi(x_k) \rightarrow \varphi_* := \inf_{x \in \mathcal{F}} \varphi(x)$, then

$$0 \in \text{co} \{ \partial f_1(\infty, u), \dots, \partial f_m(\infty, u) \} + \text{pos} \{ \cup_{i=1}^m \partial g_i(\infty; u) \} + N_{\Omega}(\infty; u). \tag{3}$$

Proof. Assume that the condition H(u) holds and there is a sequence $x_k \xrightarrow{\mathcal{F}, u} \infty$ such that $\varphi(x_k) \rightarrow \varphi_* := \inf_{x \in \mathcal{F}} \varphi(x)$.

Using an argument similar as in the proof of Theorem 3.1 and since $f_i, i \in I$, are Lipschitz at infinity in direction u , then these functions are locally Lipschitz with the same constant $L_f > 0$ for all k large enough and so the minimax function φ is Lipschitz at infinity the direction u . By Theorem 2.1, we have

$$0 \in \partial\varphi(\infty, u) + N_{\mathcal{F}}(\infty, u). \tag{4}$$

By Theorem 3.1, one has

$$\partial\varphi(\infty, u) \subseteq \text{co}\{\partial f_1(\infty, u), \dots, \partial f_m(\infty, u)\}. \tag{5}$$

Applying Proposition 2.2 with the functions $h_j = 0$, for all $j \in J$, the set S is replaced by \mathcal{F} and the condition $H(u)$ equivalent the condition (LCQ_u^∞) , then one has

$$N_{\mathcal{F}}(\infty, u) \subset \text{pos}\{\bigcup_{i=1}^m \partial g_i(\infty, u)\} + N_{\Omega}(\infty, u). \tag{6}$$

From (4), (5), (6), we obtain the inclusion (3). The proof is complete. \square

Theorem 3.3. If, for each $u \in \Omega^\infty \cap \mathbb{S}$, the condition $H(u)$ holds and

$$0 \notin \text{co}\{\partial f_1(\infty, u), \dots, \partial f_m(\infty, u)\} + \text{pos}\left\{\bigcup_{i=1}^m \partial g_i(\infty, u)\right\} + N_{\Omega}(\infty, u).$$

then $\text{sol}(\text{MNP})$ is nonempty and compact.

Proof. The conclusion is a direct consequence of Theorem 3.2 and Theorem 2.2. \square

The following example can serve as an illustration for Theorem 3.3.

Example 3.1. Let $\Omega = \mathbb{R}$ and let $f_1, f_2, g : \mathbb{R} \rightarrow \mathbb{R}$ be l.s.c functions defined by

$$f_1(x) = \begin{cases} -x - \ln(x), & \text{if } x > 0, \\ -2x, & \text{otherwise,} \end{cases}$$

$$f_2(x) = -3x,$$

$$g(x) = \begin{cases} -x + 1, & \text{if } x \leq 0, \\ -2x + 1, & \text{otherwise.} \end{cases}$$

An easy computation shows that $\Omega^\infty \cap \mathbb{S} = \{-1, 1\}$. We have $N_{\Omega}(\infty, u) = \{0\}$ for all $u \in \{-1, 1\}$. By direct computation, for all $u \in \{-1, 1\}$ then $\partial^\infty f_1(\infty, u) = \partial^\infty f_2(\infty, u) = \partial^\infty g(\infty, u) = \{0\}$. By Proposition 2.1 implies that f_1, f_2, g are Lipschitz at infinity in direction u .

We consider the following cases:

Case 1: $u = 1$. We have $\partial f_1(\infty, u) = \{-1\}, \partial f_2(\infty, u) = \{-3\}, \partial g(\infty, u) = \{-2\}$. So, the conditions of Theorem 3.3 are satisfied.

Case 2: $u = -1$. We also have $\partial f_1(\infty, u) = \{-2\}, \partial f_2(\infty, u) = \{-3\}, \partial g(\infty, u) = \{-1\}$. So, the conditions of Theorem 3.3 are satisfied.

By Theorem 3.3, $\text{sol}(\text{MNP})$ is nonempty and compact.

To the end of this subsection, we discuss several applications for the following vector optimization problem. Let us consider the problem:

$$\text{Min}_{\mathbb{R}_+^m} \{f(x) := (f_1(x), \dots, f_m(x)) \mid x \in \mathcal{F}\} \tag{VOP}$$

where f, g, \mathcal{F} is defined above.

Definition 3.1. Let $\bar{y} \in \text{clf}(\mathcal{F})$. We say that

(i) \bar{y} is a weak Pareto optimal value of (VOP) if

$$f(\mathcal{F}) \cap (\bar{y} - \text{int}\mathbb{R}_+^m) = \emptyset.$$

The set of all weak Pareto values of (VOP) is denoted by $\text{val}^W(\text{VOP})$.

(ii) \bar{y} is a Pareto optimal value of (VOP) if

$$f(\mathcal{F}) \cap (\bar{y} - \mathbb{R}_+^m \setminus \{0\}) = \emptyset.$$

The set of all Pareto values of (VOP) is denoted by $\text{val}(\text{VOP})$

(iii) $\bar{x} \in \mathcal{F}$ is a weak Pareto solution of (VOP) if

$$f(\mathcal{F}) \cap (f(\bar{x}) - \text{int}\mathbb{R}_+^m) = \emptyset.$$

The set of all weak Pareto solutions of (VOP) is denoted by $\text{sol}^W(\text{VOP})$.

(iv) $\bar{x} \in \mathcal{F}$ is a Pareto solution of (VOP) if

$$f(\mathcal{F}) \cap (f(\bar{x}) - \mathbb{R}_+^m \setminus \{0\}) = \emptyset.$$

The set of all Pareto solutions of (VOP) is denoted by $\text{sol}(\text{VOP})$.

We have a theorem to say about the necessary optimality condition for a weak Pareto value at infinity of problem (VOP).

Theorem 3.4. Let any $u \in \Omega^\infty \cap \mathbb{S}$ and $\bar{y} \in \text{val}^W(\text{VOP})$. Assume that the condition $H(u)$ is satisfied. If there is a sequence $x_k \xrightarrow{\mathcal{F}, u} \infty$ such that $f(x_k) \rightarrow \bar{y}$, then there exist $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m \setminus \{0\}$ and $\beta := (\beta_1, \dots, \beta_p) \in \mathbb{R}_+^p$ such that

$$0 \in \sum_{i \in I} \alpha_i \partial f_i(\infty, u) + \sum_{j \in J} \beta_j \partial g_j(\infty, u) + N_\Omega(\infty, u).$$

Proof. The theorem is proven by using similar arguments in the proof of [17, Theorem 4.1] and results from Theorem 3.2. □

Theorem 3.5. Assume that the condition $H(u)$ is satisfied for all $u \in \Omega^\infty \cap \mathbb{S}$ and $\text{val}^W(\text{VOP}) \neq \emptyset$. If

$0 \notin \text{co}\{\partial f_1(\infty, u), \dots, \partial f_m(\infty, u)\} + \text{pos}\{\partial g_1(\infty, u), \dots, \partial g_p(\infty, u)\} + N_\Omega(\infty, u)$ for all $u \in \Omega^\infty \cap \mathbb{S}$, then $\text{sol}^W(\text{VOP})$ is nonempty and compact.

Proof.

Claim 1: $\text{sol}^W(\text{VOP}) \neq \emptyset$.

Let $\bar{y} \in \text{val}^W(\text{VOP})$ (since $\text{val}^W(\text{VOP}) \neq \emptyset$). By the definition of the set $\text{val}^W(\text{VOP})$, $\bar{y} \in \text{cl}f(\mathcal{F})$ then there exists a sequence $x_k \in \mathcal{F}$ such that $f(x_k) \rightarrow \bar{y}$. By assumptions of the theorem and [17, Theorem 4.1], the sequence (x_k) is bounded (if otherwise, $x_k \rightarrow \infty$, then by [17, Theorem 4.1], we have $0 \in \text{co}\{\partial f_1(\infty, u), \dots, \partial f_m(\infty, u)\} + \text{pos}\{\partial g_1(\infty, u), \dots, \partial g_p(\infty, u)\} + N_\Omega(\infty, u)$, which is a contradiction). So we may assume that $x_k \rightarrow \bar{x} \in \mathcal{F}$. By the lower semicontinuity of function f_i at \bar{x} , we have

$$f_i(\bar{x}) \leq \liminf_{x \rightarrow \bar{x}} f_i(x) \leq \lim_{k \rightarrow \infty} f_i(x_k) = \bar{y}_i, \forall i \in I \Rightarrow f(\bar{x}) \in \bar{y} - \mathbb{R}_+^m.$$

We show that $\bar{x} \in \text{sol}^W(\text{VOP})$. Indeed, if otherwise, then there exists $x \in \mathcal{F}$ such that $f(x) \in f(\bar{x}) - \text{int}\mathbb{R}_+^m$. This implies that $f(x) \in f(\bar{x}) - \text{int}\mathbb{R}_+^m \subset (\bar{y} - \mathbb{R}_+^m) - \text{int}\mathbb{R}_+^m \subset \bar{y} - \text{int}\mathbb{R}_+^m$, which contradicts the fact that $\bar{y} \in \text{val}^W(\text{VOP})$. So $\text{sol}^W(\text{VOP}) \neq \emptyset$.

Claim 2: $\text{sol}^W(\text{VOP})$ is bounded.

On the contrary, there exists a sequence $x_k \in \text{sol}^W(\text{VOP})$ such that $x_k \rightarrow \infty$, as $k \rightarrow \infty$. Using similar arguments applied in the proof of Theorem 4.1, for each k we have

$$x_k \in \text{argmin}_{\mathcal{F}} \left[\max_{i \in I} (f_i(x) - f_i(x_k)) \right].$$

By [11, Corollary 6.6], for k large enough, there exist $\alpha_k^i \geq 0, \beta_k^j \geq 0$, and $u_k^i \in \partial f_i(x_k), v_k^j \in \partial g_j(x_k), w_k \in N_{\Omega}(x_k)$ such that

$$\sum_{i \in I} \alpha_k^i + \sum_{j \in J} \beta_k^j = 1, \tag{7}$$

and

$$\sum_{i \in I} \alpha_k^i u_k^i + \sum_{j \in J} \beta_k^j v_k^j + w_k = 0. \tag{8}$$

By the Lipschitzness at infinity of f_i and g_i , then sequences u_k^i, v_k^j are bounded for all k large enough. By passing to subsequence, if necessary, we can assume that $\alpha_k^i \rightarrow \alpha_i, \beta_k^j \rightarrow \beta_j$ and $u_k^i \rightarrow u_i \in \partial f_i(\infty, u), v_k^j \rightarrow v_j \in \partial g_j(\infty, u)$ such that $\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j = 1$ (by (7)). By (8), we have w_k is bounded, so $w_k \rightarrow w \in N_{\Omega}(\infty, u)$. Hence,

$$\sum_{i \in I} \alpha_i u_i + \sum_{j \in J} \beta_j v_j + w = 0. \tag{9}$$

If $\alpha_i = 0, \forall i \in I$, then by (9), we have $0 \in \sum_{j \in J} \beta_j \partial g_j(\infty, u) + N_{\Omega}(\infty, u)$ (where $\sum_{j \in J} \beta_j = 1$), which contradicts to $H(u)$ holds at ∞ in direction u . Hence, $\sum_{i \in I} \alpha_i > 0$ and so

$$0 \in \text{co}\{\partial f_1(\infty, u), \dots, \partial f_m(\infty, u)\} + \text{pos}\{\partial g_1(\infty, u), \dots, \partial g_p(\infty, u)\} + N_{\Omega}(\infty, u),$$

a contradiction.

Claim 3: $\text{sol}^W(\text{VOP})$ is closed.

Let $x_k \in \text{sol}^W(\text{VOP}), x_k \rightarrow \bar{x}$. We claim that $\bar{x} \in \text{sol}^W(\text{VOP})$. If otherwise, i.e., $\bar{x} \notin \text{sol}^W(\text{VOP})$, then there exists $x' \in \mathcal{F}$ such that

$$f_i(x') \leq f_i(\bar{x}), \forall i \in I. \tag{10}$$

By f_i l.s.c at \bar{x} , we have $f_i(\bar{x}) \leq \liminf_{x \rightarrow \bar{x}} f_i(x) \leq \liminf_{k \rightarrow \infty} f_i(x_k), \forall i \in I$. This and (10), we have $f_i(x') \leq f_i(x_k), \forall i \in I$, for all k large enough. Then, $f(x') \in f(x_k) - \text{int}\mathbb{R}_+^m$, for all k large enough. Which contradicts $x_k \in \text{sol}^W(\text{VOP})$. So $\text{sol}^W(\text{VOP})$ is closed. The proof is complete. \square

Example 3.2. We consider the problem (VOP), with the set $\Omega = \mathbb{R}$ and the functions $f_1, f_2, g : \mathbb{R} \rightarrow \mathbb{R}$ are l.s.c functions defined by

$$f_1(x) = \begin{cases} -x - \ln(x), & \text{if } x > 0, \\ -2x, & \text{otherwise,} \end{cases}$$

$$f_2(x) = -3x,$$

$$g(x) = \begin{cases} -x + 1, & \text{if } x \leq 0, \\ -2x + 1, & \text{otherwise.} \end{cases}$$

Using the results of Example 3.1, we obtain that $\text{sol}(\text{VOP})$ is nonempty and compact.

4. Conclusions

In this paper, by the concepts of directional normal cones at infinity together with directional limiting and singular subdifferentials at infinity introduced in [22], we propose several applications, including

directional optimality conditions at infinity, the nonemptiness and compactness of the global solution set. Illustrative examples were provided to highlight the effectiveness of the proposed framework.

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